Sobre la estabilización interna de la ecuación de Kuramoto-Sivashinsky

Felipe Labra Irribarra

XCI Encuentro Anual de la Sociedad de Matemática Chilena.

Trabajo conjunto con Patricio Guzmán (Universidad Técnica Federico Santa María)
y Hugo Parada (Institute de Mathématiques de Toulouse)

19 de Diciembre de 2023

Contents

1. About the equation
2. Internal stabilization linear KS: Scalar control
3. Internal stabilization linear KS: Saturated control
4. Bibliography

Contents

1. About the equation
2. Internal stabilization linear KS: Scalar control
3. Internal stabilization linear KS: Saturated control
4. Bibliography

About the equation

The Kuramoto-Sivashinsky equation is a non linear parabolic equation. In this presentation we will consider the 1-dimensional equation

$$z_t + z_{xxxx} + \lambda z_{xx} + z z_x = 0,$$

where $\lambda > 0$ is the anti-diffusion parameter. The equation was developed independently by

- Kuramoto on 1975 as a model for reaction-diffusion phenomenoms
- Sivashinsky on 1977 as a model for laminar flame front.

Contents

1 About the equation

2 Internal stabilization linear KS: Scalar control

3 Internal stabilization linear KS: Saturated control

4 Bibliography

Scalar control problem

We will consider the system

$$\begin{cases} w_t + w_{xxxx} + \lambda w_{xx} = u, & t > 0,\ x \in (0, L), \\ w(t, 0) = w(t, L) = 0, & t > 0, \\ w_x(t, 0) = w_x(t, L) = 0, & t > 0, \\ w(0, x) = w_0(x), & x \in (0, L). \end{cases} \tag{1}$$

where $w_0 \in L^2(0, L)$. We want to find a control u such that

$$||w||_{L^2(0,L)} \leq e^{-\omega t} ||w_0||_{L^2(0,L)}, \qquad \forall t \geq 0,$$

for $\omega > 0$.

Spectral Analysis

Consider the spacial operator associated to (1) given by

$$\begin{cases} A : \mathcal{D}(A) \subset L^2(0,L) \to L^2(0,L), \\ \mathcal{D}(A) = H^4(0,L) \cap H_0^2(0,L), \\ Aw = -w'''' - \lambda w''. \end{cases} \tag{2}$$

A is self-adjoint with compact resolvent. The spectrum of A is a discrete subset $\{\sigma_j\}_{j\in\mathbb{N}}$ of $\mathbb{R}$ satisfying $\lim_{j\to\infty} \sigma_j = -\infty$.

The eigenfunctions $\{e_j\}_{j\in\mathbb{N}}$ form a basis of $L^2(0,L)$:

$$\begin{cases} -e_j'''' - \lambda e_j'' = \sigma_j e_j \\ e_j(0) = e_j(L) = e_j'(0) = e_j(L) = 0. \end{cases} \tag{3}$$

Spectral Analysis

Important: The operator A has some positive eigenvalues if and only if $\lambda > 4\pi^2$.

Let $\omega > 0$ such that

$$\begin{cases} \sigma_j \geq 0, & \forall j \in \{1, ..., n\}, \\ -\sigma_j > \omega > 0, & \forall j \geq n+1. \end{cases} \tag{4}$$

Idea: Find a control $u(t, x)$ written as

$$u(t, x) = \sum_{j=1}^{n} G_j(t) e_j(x)$$

which affects only the unstable eigenvalues. We want to find the control gains $G_j(t)$.

Expansion of the solution

Due to the spectral analysis, we have that

$$w(t,x) = \sum_{j=1}^{\infty} w_j(t) e_j(x). \tag{5}$$

and

$$E(t) = \frac{1}{2}\int_0^L |w|^2 dx = \frac{1}{2}\sum_{j=1}^{\infty} w_j^2(t).$$

the objective is to prove

$$E(t) \leq e^{-2\omega t} E(0), \qquad \forall t \geq 0.$$

Construction of the control

Multiplying (1) by w and integrating in $(0, L)$, we get that

$$\frac{1}{2}\frac{d}{dt}\left(\int_0^L |w|^2 dx\right) + \int_0^L -w(u - w_{xxxx} - \lambda w_{xx})dx = 0. \tag{6}$$

Using the spectral analysis again, we find that

$$\begin{aligned}\int_0^L -w(u - w_{xxxx} - \lambda w_{xx})dx &= \int_0^L -w(A + u)dx \\ &= \sum_{j=1}^{\infty} -\sigma_j w_j^2(t) + \sum_{j=1}^{n} -w_j(t)G_j(t).\end{aligned}$$

Exponential stability

In order to use the sign of the eigenvalues, we note that

$$\int_0^L -w(A+u)dx = \sum_{j=n+1}^{\infty} -\sigma_j w_j^2(t) + \sum_{j=1}^{n} \left(-\sigma_j w_j^2(t) - w_j(t)G_j(t)\right)$$

For the blue part, we use that $-\sigma_j > \omega > 0,\ \forall j \geq n+1$ to obtain

$$\sum_{j=n+1}^{\infty} -\sigma_j w_j^2(t) > \omega \sum_{j=n+1}^{\infty} w_j^2(t). \tag{7}$$

Exponential stability

For the red part, we need that

$$G_j(t) = -(\omega + \sigma_j) w_j(t), \qquad \forall j \in \{1, ..., n\}. \tag{8}$$

In this case, we find that

$$\sum_{j=1}^{n} \left(-\sigma_j w_j^2(t) - w_j(t) G_j(t)\right) = \omega \sum_{j=1}^{n} w_j^2(t)$$

Therefore, using the two previous estimations we find out that under the control

$$u(t,x) = \sum_{j=1}^{n} (-\omega - \sigma_j) w_j(t) e_j(x), \tag{9}$$

Exponential stability

we get the estimation

$$\int_0^L -w(A+u)dx \geq \omega \sum_{j=n+1}^{\infty} w_j^2(t) + \omega \sum_{j=1}^{n} w_j^2(t) = \omega \sum_{j=1}^{\infty} w_j^2(t) = 2\omega E(t).$$

Hence we get that

$$0 > \dot{E}(t) + 2\omega E(t)$$

where we deduce

$$E(t) \leq e^{-2\omega t} E(0), \qquad \forall t \geq 0,$$

or equivalently

$$||w||_{L^2(0,L)} \leq e^{-\omega t} ||w_0||_{L^2(0,L)}, \qquad \forall t \geq 0.$$

Principal result

Theorem

Given a solution w of (1) with initial value $w_0 \in L^2(0, L)$ and a decay rate $\omega > 0$. Then there exists a closed feedback law

$$u(t, x) = \sum_{j=1}^{n} (-\omega - \sigma_j) w_j(t) e_j(x), \tag{10}$$

which exponentially stabilizes the system (1) with decay rate ω. Equivalently, the following inequality holds

$$||w(t, \cdot)||_{L^2(0,L)} \leq e^{-\omega t} ||w_0||_{L^2(0,L)}, \qquad t \geq 0, \tag{11}$$

where $\{\sigma_j\}_{j \in \mathbb{N}}$ and $\{e_j\}_{j \in \mathbb{N}}$ denote the eigenvalues and eigenfunctions given by the spectral analysis.

Contents

1 About the equation

2 Internal stabilization linear KS: Scalar control

3 Internal stabilization linear KS: Saturated control

4 Bibliography

Problem formulation

We will study the problem

$$\begin{cases} w_t = -w_{xxxx} - \lambda w_{xx} + v, & t > 0,\ x \in (0, L), \\ w(t,0) = w(t,L) = 0, & t > 0, \\ w_x(t,0) = w_x(t,L) = 0, & t > 0, \\ w(0,x) = w_0(x), & x \in (0,L). \end{cases} \tag{12}$$

where

$$v = \sum_{k=1}^{m} b_k(x) \mathfrak{sat}(u_k(t))$$

and given $\ell \geq 0$ we have

$$\mathfrak{sat}(u) = \begin{cases} -\ell, & u \leq -\ell \\ u, & -\ell \leq u \leq \ell \\ \ell, & \ell \leq u \end{cases}$$

Problem formulation

Given a control $u : (0, +\infty) \to \mathbb{R}^m$, the puntual saturation is

$$\mathfrak{sat}(u(t)) = \begin{bmatrix} \mathfrak{sat}(u_1(t)) \\ \vdots \\ \mathfrak{sat}(u_m(t)) \end{bmatrix},$$

and consider

$$\mathbf{b}(x) = (b_1, ..., b_m)$$

where $b_k : [0, L] \to \mathbb{R}$ are the localizations of the u_k component. Following the previous steps, we find

$$\dot{w}_j(t) = \sigma_j w_j(t) + \mathbf{b}_j \cdot \mathfrak{sat}(u(t)), \qquad \forall j \in \mathbb{N}, \tag{13}$$

where $\mathbf{b}_j = (b_{j1}, ..., b_{jm}) \in \mathbb{R}^{1\times m}$ for every $j \in \mathbb{N}$.

Problem formulation

From the spectral analysis, we take $\eta > 0$ such that

$$\begin{cases} \sigma_j < -\eta < 0, & \forall j \geq n+1, \\ \sigma_j > 0, & \forall j \leq n. \end{cases} \tag{14}$$

therefore, using the notation

$$z := \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix}, \quad \mathbf{A} := \begin{bmatrix} \sigma_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \sigma_n \end{bmatrix}, \quad \mathbf{B} := \begin{bmatrix} b_{11} & \cdots & b_{1m} \\ \vdots & \ddots & \vdots \\ b_{n1} & \cdots & b_{nm} \end{bmatrix}, \tag{15}$$

we find the finite dimensional system

$$\dot{z}(t) = \mathbf{A}z(t) + \mathbf{B}\mathfrak{sat}(u(t)). \tag{16}$$

Saturation induces a coupling between the finite system and the infinite system!

Feedback law

Let X_n be the subspace of $L^2(0, L)$ generated by $\{e_j\}_{j=1}^n$. Take $K : X_n \to \mathbb{R}^m$ and consider the control

$$u(t) = \mathbf{K}z(t) \tag{17}$$

then we have the system

$$\dot{w}_j(t) = \sigma_j w_j(t) + \mathbf{b}_j \cdot \mathfrak{sat}(\mathbf{K}z(t)), \qquad j \in \mathbb{N}, \tag{18}$$

where the first $n \in \mathbb{N}$ equations can be written as

$$\dot{z}(t) = \mathbf{A}z(t) + \mathbf{B}\mathfrak{sat}(\mathbf{K}z(t)). \tag{19}$$

Cascade system

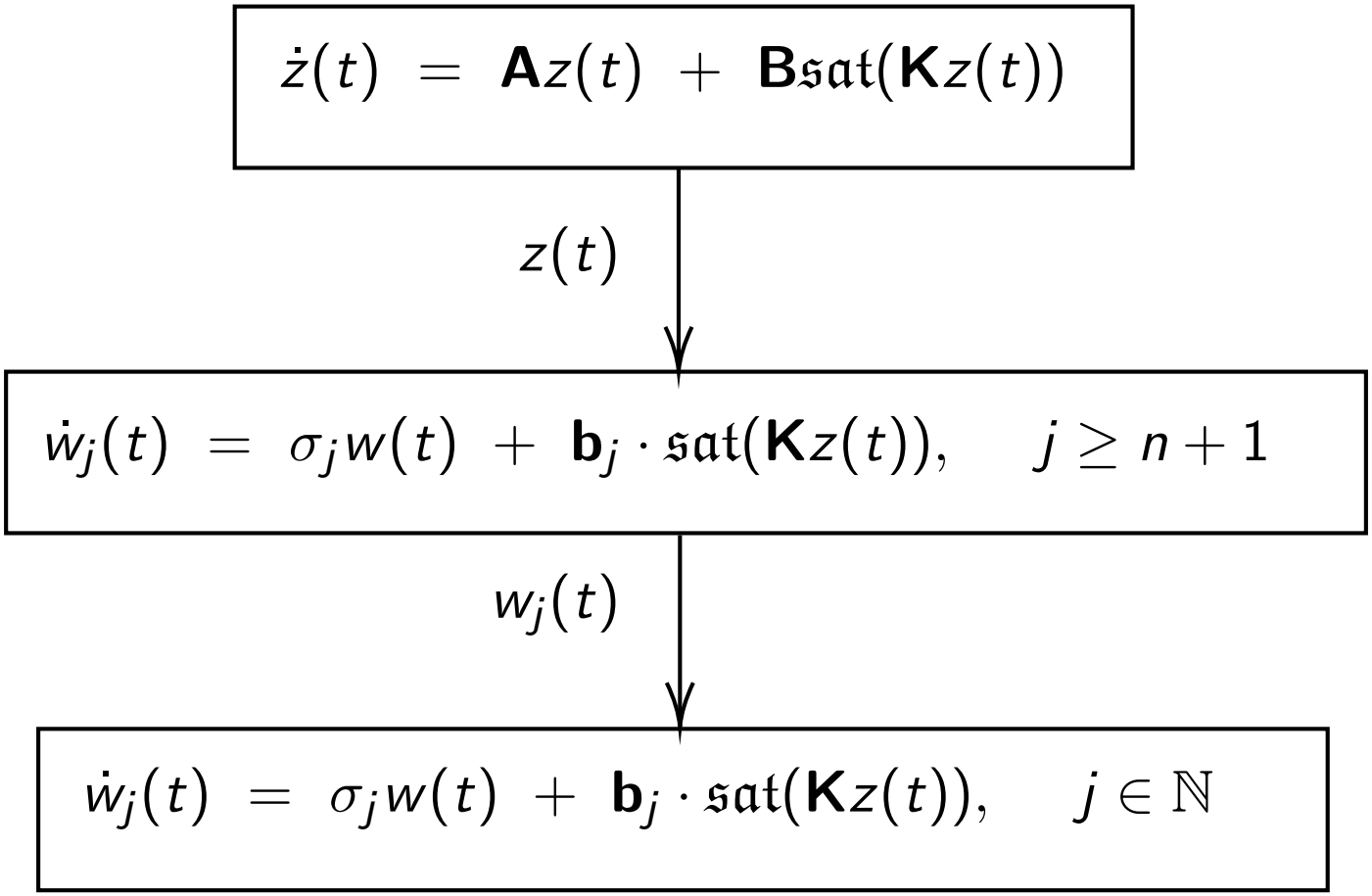


Figura: Cascade system justification

Region of attraction

Definition

$S \subset \mathbb{R}^n$ is a region of attraction if

1. $0 \in \operatorname{int} S$;
2. *For every initial condition $z_0 \in S$ we have that $z(t) \to 0$ if $t \to \infty$;*
3. *For every initial condition $z_0 \in S$, it follows that $z(t) \in S$ for all $t \geq 0$.*

Denote by $X_n^{\perp}$ the orthogonal complement of X_n and

$$\ell^2_{j>n}(\mathbb{R}) = \left\{ \{x_k\}_{k\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}} : \quad x_k = 0,\ \forall k \in \{1, ..., n\} \wedge \sum_{k=n+1}^{\infty} |x_k|^2 < \infty \right\}$$

Region of attraction and stabilization

Proposition

Chosing $\mathbf{K}$ *such that the finite dimensional system is locally exponentially stable in* $0 \in \mathbb{R}^n$ *with region of attraction* $S \in \mathbb{R}^n$*. Then:*

1. *System* (18) *is locally exponentially stable in 0 with region of attraction* $S \times \ell^2_{j>n}(\mathbb{R})$.
2. *System* (12) *is locally exponentially stable in 0 with region of attraction* $i(S) \times X_n^{\perp}$.

In addition, for any closed and bounded set $G \subseteq \text{int}(i(S) \times X_n^{\perp})$*, there exist two positive values* M *and asuch that for any initial condition* $w(0,\cdot) \in G$*, the solution* $w(\cdot)$ *satisfies*

$$||w(t,\cdot)||_{L^2(0,L)} \leq Me^{-at} ||w(0,\cdot)||_{L^2(0,L)}, \quad t \geq 0.$$

Idea behind the proposition

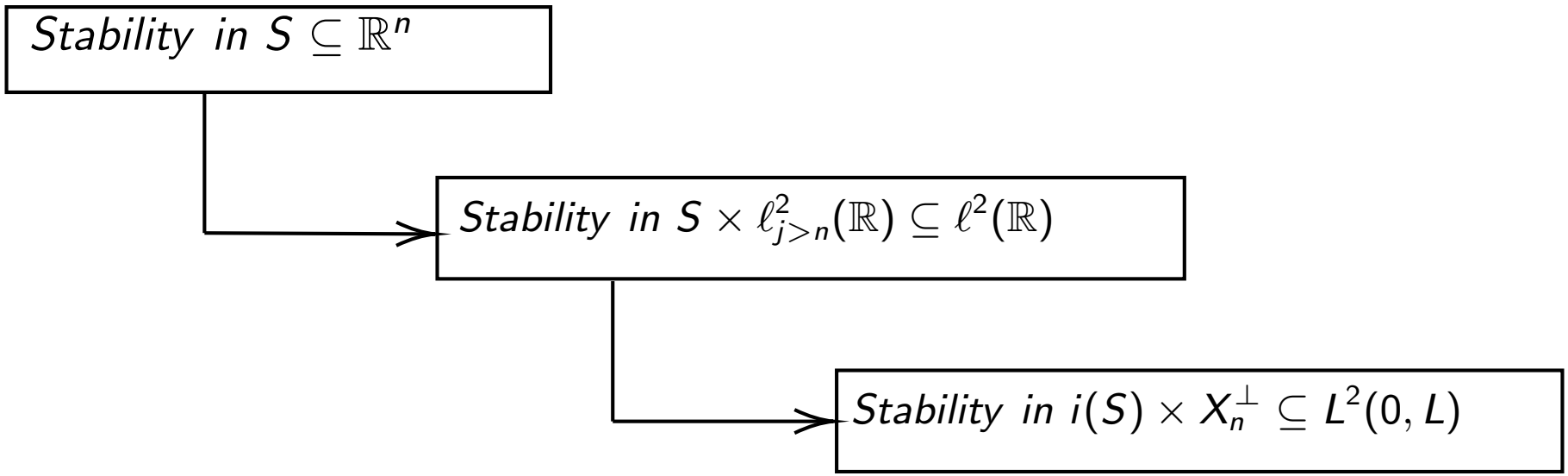


Figura: Relationship with the cascade system

Manipulating saturations

We define the *deadzone nonlinearity* $\phi : \mathbb{R}^m \to \mathbb{R}^m$

$$\phi(u) = \mathfrak{sat}(u) - u, \quad \forall u \in \mathbb{R}^m,$$

Also, the following *generalized sector condition* holds

Lemma

If for some $\mathbf{C}, \mathbf{K} \in \mathbb{R}^{m \times n}$, $z \in \mathbb{R}^n$ *and* $j \in \{1, ..., m\}$ *it holds that* $|((\mathbf{K} - \mathbf{C})z)_j| \leq \ell$, *then*

$$\phi_j(\mathbf{K}z)(\phi_j(\mathbf{K}z) + (\mathbf{C}z)_j) \leq 0.$$

Then we can prove

Proposition

Let $\mathbf{K} \in \mathbb{R}^{m\times n}$ *such that* $\mathbf{A}+\mathbf{BK}$ *is Hurwitz. Let* $\mathbf{P} \in \mathbb{R}^{n\times n}$ *be symmetric positive definite,* $\mathbf{D} \in \mathbb{R}^{m\times m}$ *diagonal positive definite and* $\mathbf{C} \in \mathbb{R}^{m\times n}$ *such that*

$$\mathbf{M}_1 := \begin{pmatrix} (\mathbf{A}+\mathbf{BK})^\top \mathbf{P} + \mathbf{P}(\mathbf{A}+\mathbf{BK}) & \mathbf{PB} - (\mathbf{DC})^\top \\ (\mathbf{PB})^\top - \mathbf{DC} & -2\mathbf{D} \end{pmatrix} \prec 0, \tag{20}$$

and

$$\mathbf{M}_2 := \begin{pmatrix} \mathbf{P} & (\mathbf{K}-\mathbf{C})^\top \\ \mathbf{K}-\mathbf{C} & \ell^2 \mathbf{I}_m \end{pmatrix} \succeq 0. \tag{21}$$

Then the finite-dimensional system (16) *is locally exponentially stable in 0 with a region of attraction given by*

$$\mathcal{A} := \{z \in \mathbb{R}^n : \quad z^\top \mathbf{P} z \leq 1\}.$$

Proposition

Moreover, in $\mathcal{A}$ *the function* $V_1(z) := z^\top \mathbf{P} z$ *with* $z \in \mathbb{R}^n$ *decreases exponentially fast to* 0 *along the solutions to* (16)*, i.e, there is a constant* $\alpha > 0$ *so that*

$$\dot{V}_1(z) \leq -\alpha |z|^2, \quad z \in \mathcal{A}.$$

The principal result of this part follows from the previous propositions.

Theorem (Stabilization in $L^2(0, L)$)

Consider system (12) *with* $(\mathbf{A}, \mathbf{B})$ *stabilizable. Let* $\mathbf{K}$ *such that* $\mathbf{A} + \mathbf{BK}$ *is Hurwitz and* $K : X_n \to \mathbb{R}^m$ *be as* (17)*. Then the closed-loop system with feedback* K

$$\begin{cases} w_t(t,x) = -w_{xxxx}(t,x) - \lambda w_{xx} + u, & t > 0,\ x \in (0,L), \\ w(t,0) = w(t,L) = 0, & t > 0 \\ w_x(t,0) = w_x(t,L) = 0, & t > 0 \\ w(0,x) = w_0(x), & x \in (0,L). \end{cases}$$

where

$$u = \sum_{j=1}^{m} b_k(x) \mathfrak{sat}((K\pi_n w(t,\cdot))_k)$$

is locally exponentially stable in 0 with region of attraction $i(\mathcal{A}) \times X_n^{\perp}$*. In addition, the constants of decay can be chosen uniformly on* $i(\mathcal{A}) \times X_n^{\perp}$*.*

Stabilization in $H_0^2(0,L)$.

Objective: To prove that there exists $(C,\omega) \in (1,\infty) \times (0,\infty)$ such that

$$||w||^2_{H_0^2(0,L)} \leq Ce^{-\omega t} ||w_0||^2_{H_0^2(0,L)}. \tag{22}$$

We will considerate the Lyapunov candidate function

$$V_2(t) = -\frac{1}{2}\langle Aw, w\rangle_{L^2(0,L)} + \frac{M}{2}V_1(t) = -\frac{1}{2}\sum_{j=1}^{\infty}\sigma_j w_j(t)^2 + \frac{M}{2}z^\top \mathbf{P}z,$$

where $\beta_{min}, \beta_{max} > 0$ are the min and max eigenvalues of $\mathbf{P}$.

To prove the result we will need the following lemmas:

Previous lemmas

Lemma

There are constants $C_1, C_2 > 0$ *such that*

$$\frac{C_1}{2}|z|^2 + \frac{C_1}{2C_2}||w||^2_{H_0^2(0,L)} \leq V_2(t), \qquad \forall t \geq 0. \tag{23}$$

Lemma

Exists $\omega > 0$ *such that*

$$V_2(t) \leq e^{-\omega t}\, V_2(0), \qquad \forall t \geq 0. \tag{24}$$

Lemma

Exists $C_4 > 0$ *such that*

$$V_2(0) \leq C_4||w_0||^2_{H_0^2(0,L)}. \tag{25}$$

Principal result

Theorem (Stabilization in $H_0^2(0,L)$.)

Let w solution of (12), z defined in (15) and the constants

$$C_1 = \text{mín}\left\{\frac{M\beta_{min}-\sigma_1}{2}, \frac{1}{2}\right\}, \quad C_2 = \text{máx}\left\{\lambda^2, 2-\frac{\lambda^2}{\sigma_{n+1}}\right\}, \quad C_4 = \frac{1+M\beta_{max}C_p^2}{2},$$

y

$$C_3 > \begin{cases} \frac{1}{\lfloor\alpha\rfloor}, & \alpha \geq 1, \\ \frac{1}{2}, & \alpha < 1, \end{cases}, \quad M > \text{máx}\left\{\frac{\sigma_1}{\beta_{min}}, -\frac{1}{\sigma_{n+1}}, 2C_3\beta_{max}\right\}.$$

Then in follows that

$$||w||^2_{H_0^2(0,L)} \leq \frac{2C_2C_4}{C_1} e^{-\omega t} ||w_0||^2_{H_0^2(0,L)}, \tag{26}$$

with decay rate

$$\omega = \frac{1}{2C_3\beta_{max}}.$$

Thanks for your attention!

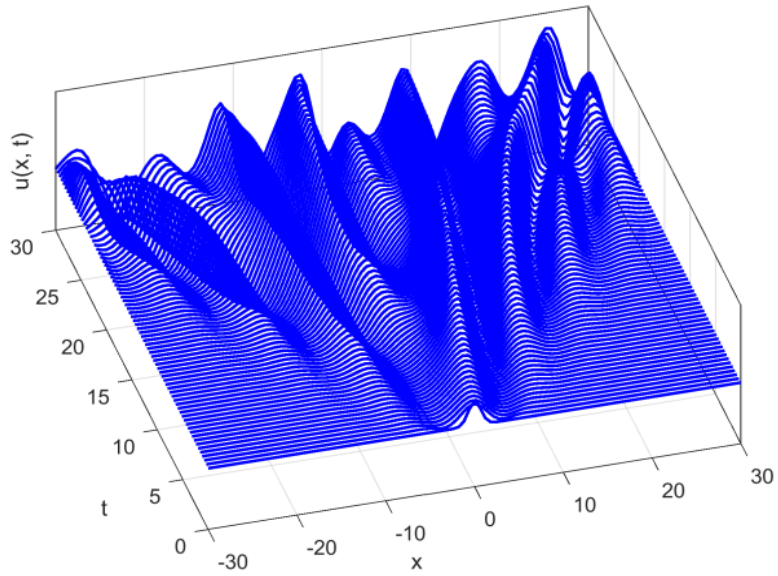


Figura: Numerical solution of non-linear KS in 3D using Runge-Kutta

H. Bhatt and A. Chowdhury (2019)
A compact fourth-order implicit-explicit Runge-Kutta type scheme for numerical solution of the Kuramoto-Sivashinsky equation.

Contents

1. About the equation
2. Internal stabilization linear KS: Scalar control
3. Internal stabilization linear KS: Saturated control
4. Bibliography

Referencias

- J. M. Coron (2007)
  Control and Nonlinearity
- M. Krstić, and A. Smyshlyaev (2008)
  Boundary control of PDEs
- W. Liu (2010)
  Elementary Feedback Stabilization of the Linear Reaction-Convection-Diffusion Equation and the Wave Equation
- H. Brezis (2011)
  Functional Analysis, Sobolev Spaces and Partial Differential Equations
- S. Salsa (2015)
  Partial Differential Equations in action: From modelling to theory