On the robustness of the stability of first-order hyperbolic systems with respect to different perturbations

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1. Transfer functions: ODEs and coupled transport
2. Second-order perturbation: Stability and robustness with boundary coupling
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5. Conclusions and further work

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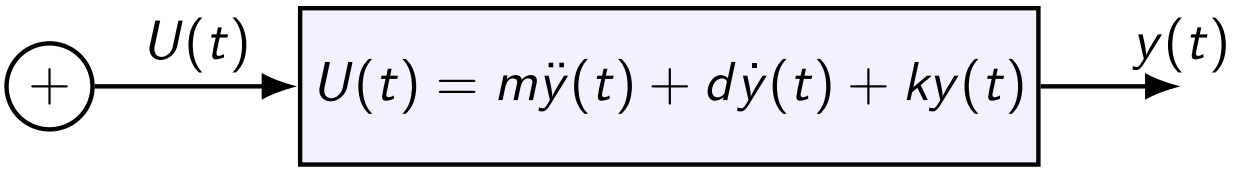
From the equation to the input-output map

Classical example: mass-spring-damper, with input $U(t)$ and output $y(t)$,

$$m\ddot{y}(t) + d\dot{y}(t) + ky(t) = U(t).$$

Laplace transform with zero initial conditions: $Y(s) = G(s)U(s)$,

$$G(s) = \frac{1}{ms^2 + ds + k}$$



The transfer function $G(s)$ replaces the differential equation by an algebraic relation. Stability is read off the location of the poles of G.

Definition (Transfer function)

Given a linear system $\dot{z} = Az + Bu$, $y = Cz + Du$, if $Y(s) = G(s)U(s)$ for $\Re(s) > \alpha$, G is the transfer function of the system.

Remark (From ODEs to PDEs)

For ODEs, $G(s)$ is usually rational. For PDEs, $G(s)$ is typically irrational with infinitely many poles and zeros as $\cosh(\sqrt{s})/\sinh(\sqrt{2s})$.

Example: transport equation with delay

$$\begin{cases} \partial_t y(t,x) + v\partial_x y(t,x) = 0, & x \in (0,1), \\ y(t,0) = U(t), \qquad Y(t) = y(t,1). \end{cases}$$

Laplace transform in time $\Rightarrow y(s,x) = y(s,0)e^{-sx/v}$, hence taking $x = 1$ we obtain $Y(s) = G(s)U(s)$ with

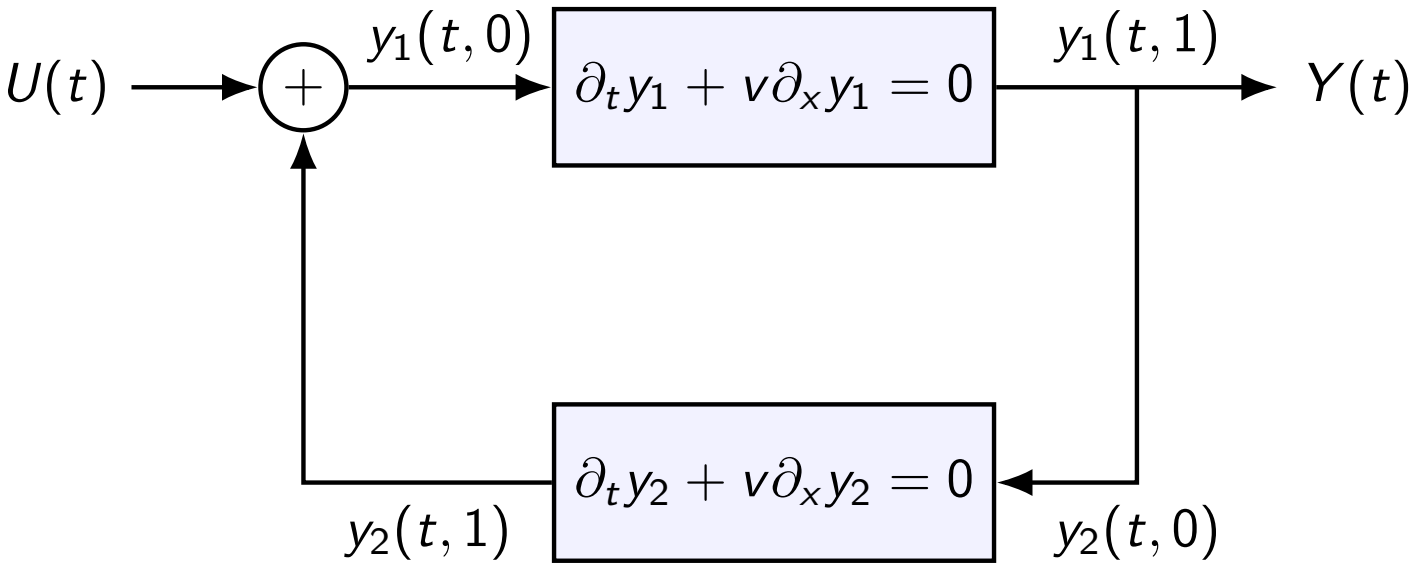
$$G(s) = e^{-s\tau}, \qquad \tau = \frac{1}{v},$$

a pure delay $Y(t) = U(t-\tau)$. This is the same relation as if we had a pure delay mechanism.

Note that if $U(t) \equiv 0$, the system is stable in finite time.

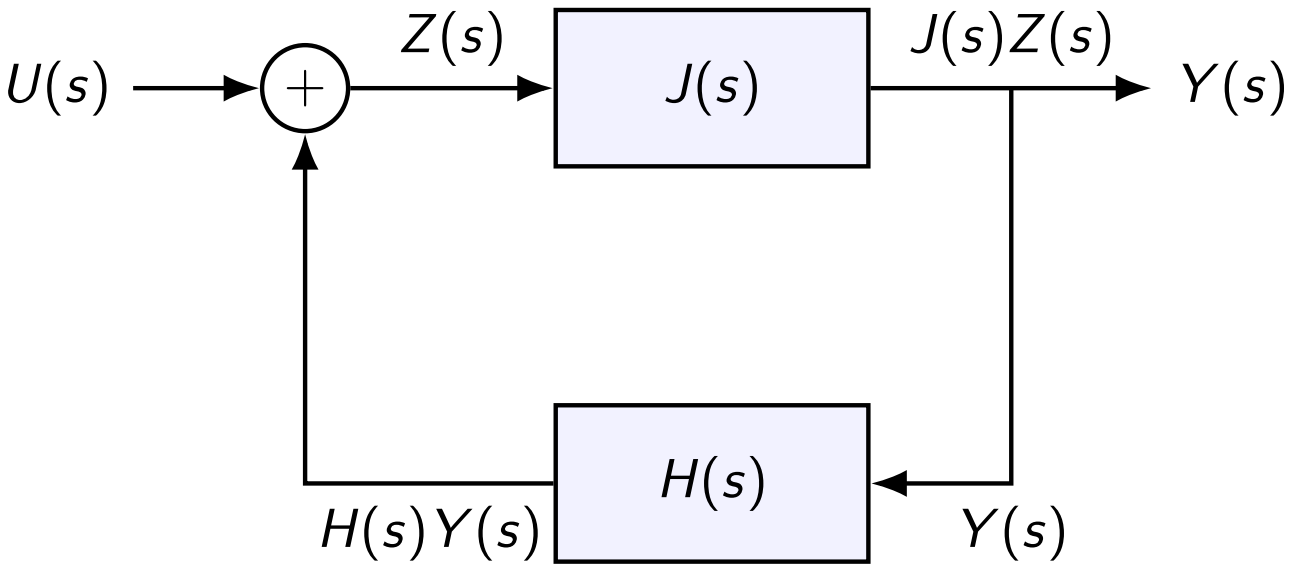
Important example: Coupled transport equations

$$\begin{cases} \partial_t y_1 + v\partial_x y_1 = 0, & \partial_t y_2 + v\partial_x y_2 = 0, \\ y_1(t,0) = y_2(t,1) + U(t), & y_2(t,0) = y_1(t,1), \\ Y(t) = y_1(t,1). \end{cases}$$



Control and observation are anti-located on the boundary. This is the main issue.

Interconnecting transfer functions



$$Y(s) = J(s)Z(s),\ Z(s) = U(s) + H(s)Y(s) \implies \frac{Y(s)}{U(s)} = \frac{J(s)}{1 - H(s)J(s)}.$$

With $J(s) = H(s) = e^{-s\tau}$,pure transport with $s \in \mathbb{C}$:

$$G(s) = \frac{Y(s)}{U(s)} = \frac{e^{s\tau}}{e^{2s\tau} - 1}$$

Unstable open-loop. Static proportional control

Poles: $e^{2s\tau} - 1 = 0 \Rightarrow s_n = \dfrac{2\pi n}{\tau} i$ (imaginary axis).

With the static control $U(t) = -2k_p Y(t)$, with k_p to be chosen, the poles lie on two vertical lines:

$$\sigma_1 = v \ln\left(-|k_p| + \sqrt{1 + k_p^2}\right) > 0, \qquad \sigma_2 < 0.$$

There are always unstable poles: no choice of k_p stabilizes the system with proportional feedback.

Delayed feedback: Stabilizes, but is not robust!

The dynamic control

$$U(t) = -2k_1 Y(t) - k_2 Y(t-\tau), \quad \mathbf{K} := \begin{pmatrix} -2k_1 & 1 & -k_2 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

can indeed stabilize (choosing suitable k_1, k_2). But robustness with respect to errors in the transport velocity requires

$$\overline{\rho}(\mathbf{K}) < 1, \qquad \text{and one shows} \qquad \overline{\rho}(\mathbf{K}) = |k_1| + \sqrt{1 + k_1^2} + |k_2| \geq 1$$

for every (k_1, k_2). In particular, the "dead-beat" control $k_1 = 0, k_2 = 1$, has $\overline{\rho}(\mathbf{K}) = \sqrt{2}$.



R. Silkowski (1976)
Star Shaped Regions of Stability in Hereditary Systems
PhD thesis, Brown University.

Can robustness be recovered by some perturbation?

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Second-order perturbation: Stability and robustness with boundary coupling

A second-order dissipative term is added to each channel ($v = 1$):

$$\begin{cases} \partial_t y_1 + \partial_x y_1 - \eta \partial_{xx}^2 y_1 = 0, \\ \partial_t y_2 + \partial_x y_2 - \eta \partial_{xx}^2 y_2 = 0, \\ y_1(t,0) = y_2(t,1) + U(t), \quad y_2(t,0) = y_1(t,1), \\ \partial_x y_1(t,1) = \partial_x y_2(t,1) = 0. \end{cases}$$



G. Bastin, J-M. Coron, A. Hayat (2023)
Diffusion and robustness of boundary feedback stabilization of hyperbolic systems
Mathematics of Control, Signals and Systems.

The transfer function

Taking Laplace transform in time for the channels we obtain $y_i(s,x) = A_i(s)e^{\lambda_1(s)x} + B_i(s)e^{\lambda_2(s)x}$ with λ_j solving $\eta\lambda^2 - \lambda - s = 0$:

$$\lambda_{1,2}(s) = \frac{1 \pm \sqrt{1+4\eta s}}{2\eta}.$$

Hence the transfer function $f_\eta(s) = y_i(s,1)/y_i(s,0)$ is

$$f_\eta(s) = \frac{\lambda_1(s) - \lambda_2(s)}{\lambda_1(s)e^{-\lambda_2(s)} - \lambda_2(s)e^{-\lambda_1(s)}}$$

Remark

As $\eta \to 0$, we recover the transport case $f_\eta(s) \to f_0(s) = e^{-s}$.

The characteristic equation

With the same interconnection argument, $G(s) = f_\eta(s)/(1 - f_\eta^2(s))$.
Closing the loop with $U(t) = -Y(t-1)$, the characteristic equation is

$$f_\eta(s)^2 - f_\eta(s)e^{-s} - 1 = 0 \tag{1}$$

We define the spectrum and maximal spectral abscissa as,

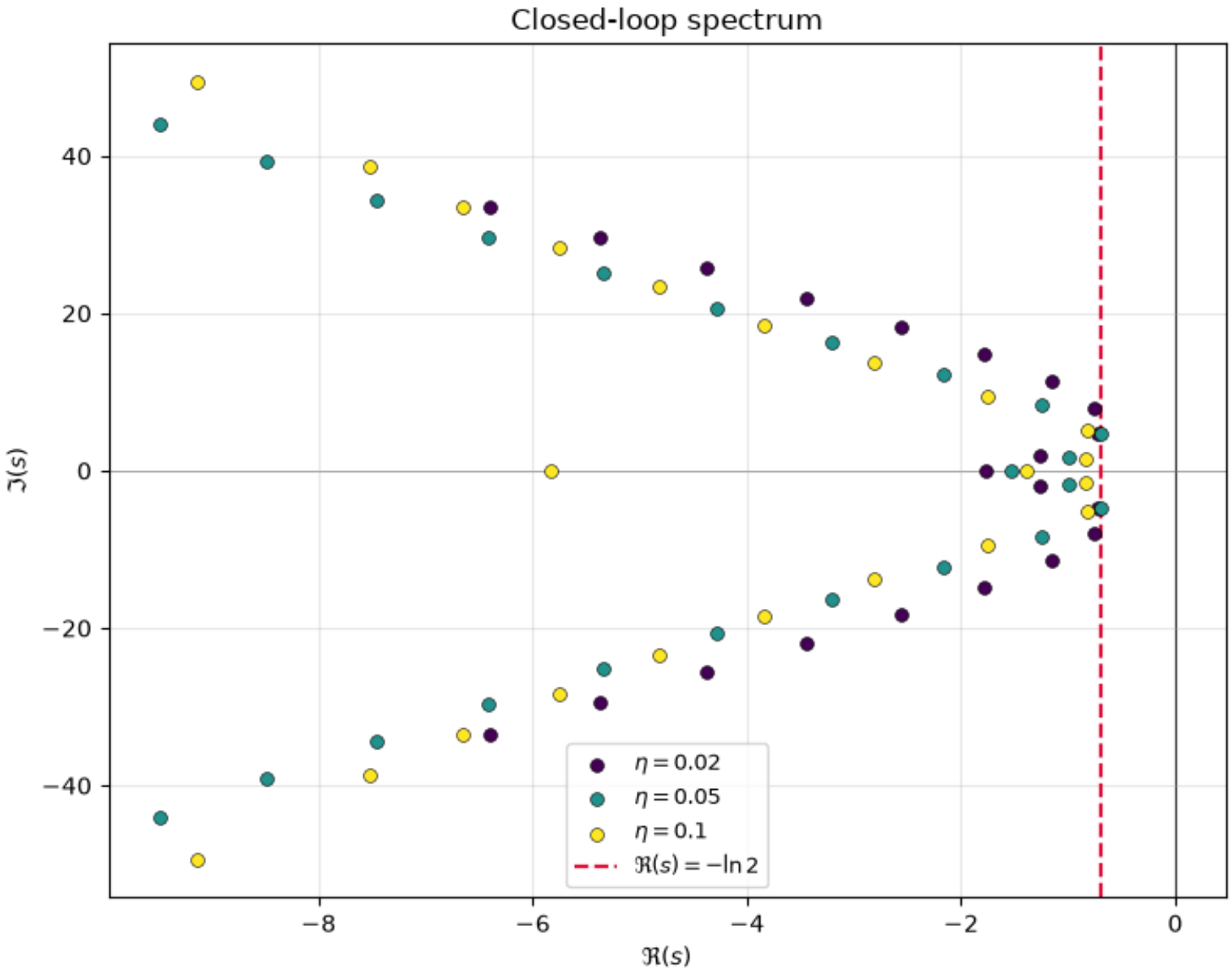
$$\mathcal{S}_\eta = \{s \in \mathbb{C} : s \text{ solves } (1)\}, \quad \sigma_\eta = \sup\{\Re(s) : s \in \mathcal{S}_\eta\}.$$

Theorem (Theorem 1, Bastin-Coron-Hayat 2023)

For every $\delta > 0$, there exists $\eta_1 > 0$ such that for all $\eta \in (0, \eta_1)$, the maximal spectral abscissa satisfies

$$\sigma_\eta \leq -\ln(2) + \delta.$$

Closed-loop spectrum for different $\eta > 0$



Is the bound ln(2) optimal?

Conjecture

The bound $\ln(2)$ *is optimal: for every* $\delta > 0, \eta_1 > 0$ *there exists* $\eta \in (0, \eta_1)$ *with*

$$-\ln(2) - \delta \leq \sigma_\eta \leq -\ln(2) + \delta.$$

Spoiler: This was proved in 2025. Wait for the next section!

Remark (Loss of continuity)

At $\eta = 0$*: stable in finite time* ($\sigma_0 = -\infty$)*. For any small* $\eta > 0$*, the uniform bound* $\ln(2)$ *appears.*

Sketch of the proof: Change of variable

Remember the characteristic equation

$$f_\eta(s)^2 - f_\eta(s)e^{-s} - 1 = 0.$$

We propose a change of variable

$$X_\eta(z) = \frac{1+z}{2}e^{-(1-z)/2\eta} - \frac{1-z}{2}e^{-(1+z)/2\eta}, \quad z := \sqrt{1+4\eta s}.$$

Hence, the characteristic equation is given by

$$X_\eta^2(z) + ze^{-(z^2-1)/4\eta}X_\eta(z) - z^2 = 0. \tag{2}$$

Denote by $\mathcal{Z}_\eta$ the set of non-zero roots of (2).

Sketch of the proof: On viscosities and poles

Now consider a sequence

$$(\eta_n)_{n\in\mathbb{N}}: \quad 0 < \eta_n \in \mathbb{R},\ \forall n \in \mathbb{N}, \qquad \lim_{n\to\infty} \eta_n = 0^+,$$

and an associated sequence

$$(s_n)_{n\in\mathbb{N}}: \quad s_n \in \mathcal{S}_{\eta_n},\ \forall n \in \mathbb{N}.$$

To prove the theorem, we study the accumulation points of $(s_n)_{n\in\mathbb{N}}$. By definition, $\overline{s}$ is an accumulation point of $(s_n)_{n\in\mathbb{N}}$ if there is a subsequence $(s_{n_k})_{k\in\mathbb{N}}$ such that $s_{n_k} \to \overline{s}$. We abuse notation and write $(s_n)_{n\in\mathbb{N}}$ even when working with the subsequence.

Sketch of the proof: Structural lemmas

Lemma (Lemma 1, Bastin-Coron-Hayat 2023)

Let $(\eta_n)_{n\in\mathbb{N}}$ be a sequence as above and let $(z_n)_{n\in\mathbb{N}}$ be the associated sequence of elements of $\mathcal{Z}_{\eta_n}$. Let $\overline{z}$ be an accumulation point of $(z_n)_{n\in\mathbb{N}}$. Then

1. $\Re(\overline{z}^2) \leq 1$. *Precisely,* $\Re(\overline{z}^2) \in [-\infty, 1]$.
2. *If* $\Re(\overline{z}^2) = 1$, *then* $(\Re(\overline{z}))^2 = 1$ *and* $\Im(\overline{z}) = 0$.

Lemma (Lemma 2, Bastin-Coron-Hayat 2023)

Let $(\eta_n)_{n\in\mathbb{N}}$ be a sequence as before and $(s_n)_{n\in\mathbb{N}}$ an associated sequence. Let $\overline{s}$ be an accumulation point of $(s_n)_{n\in\mathbb{N}}$. Then

$$\Re(\overline{s}) \leq -\ln(2), \quad \textit{precisely,}\ \Re(\overline{s}) \in [-\infty, -\ln(2)].$$

Sketch of the proof: The contradiction

Theorem (Theorem 1, Bastin-Coron-Hayat 2023)

For every $\delta > 0$, there exists $\eta_1 > 0$ such that $\eta \in (0, \eta_1)$ implies

$$\sigma_\eta \leq -\ln(2) + \delta.$$

Proof: By contradiction, for every $\delta > 0$ there exist $\eta_1 \in (0, \eta)$ and $s \in \mathcal{S}_{\eta_1}$ such that $\Re(s) > -\ln(2) + \delta$.

It follows that we can construct a sequence of diffusion parameters $(\eta_n)_{n\in\mathbb{N}}$, positive and with $\eta_n \to 0^+$, with an associated sequence $(s_n)_{n\in\mathbb{N}}$ with $s_n \in \mathcal{S}_{\eta_n}$. For this sequence, we have, by the contradiction hypothesis, that $\Re(\bar{s}) > -\ln(2) + \delta$, which contradicts Lemma 2. □

Robustness with respect to velocity error $\varepsilon > 0$

Here we study stability under $v \mapsto v + \varepsilon$. For simplicity we assume $v = 1$.

$$\begin{cases} \partial_t y_1 + (1+\varepsilon)\partial_x y_1 - \eta \partial_{xx}^2 y_1 = 0, \\ \partial_t y_2 + (1+\varepsilon)\partial_x y_2 - \eta \partial_{xx}^2 y_2 = 0, \\ y_1(t,0) = y_2(t,1) + U(t), \quad y_2(t,0) = y_1(t,1), \\ \partial_x y_1(t,1) = \partial_x y_2(t,1) = 0. \end{cases}$$

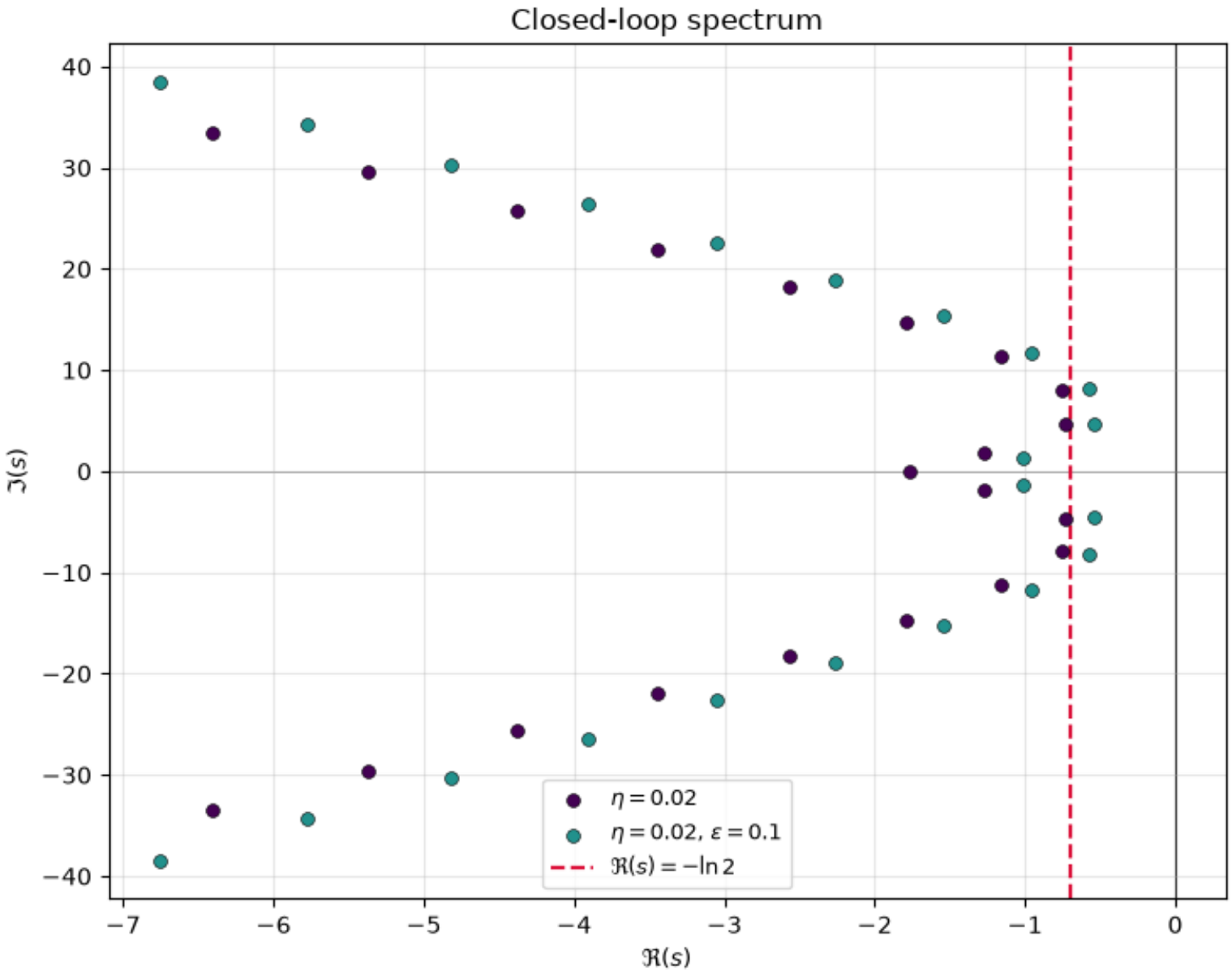
Theorem (Theorem 2, Bastin-Coron-Hayat 2022)

Let δ, η be as in Theorem 1. There exists $\varepsilon_1 > 0$ such that, for $|\varepsilon| < \varepsilon_1$,

$$\sigma_{\eta,\varepsilon} \leq -\ln(2) + 2\delta.$$

Important: Same calculations for $f_{\eta,\varepsilon}$ and $\lambda_{1,2}$!

Perturbed closed-loop spectrum: $U(t) = -Y(t-1)$



Sketch of the proof

By contradiction we construct a sequence $(s_n)_{n\in\mathbb{N}}$ such that

$$\Re(s_n) > -\ln(2) + 2\delta, \quad s_n \in \mathcal{S}_{\eta,\varepsilon_n},\ \forall n \in \mathbb{N}. \tag{3}$$

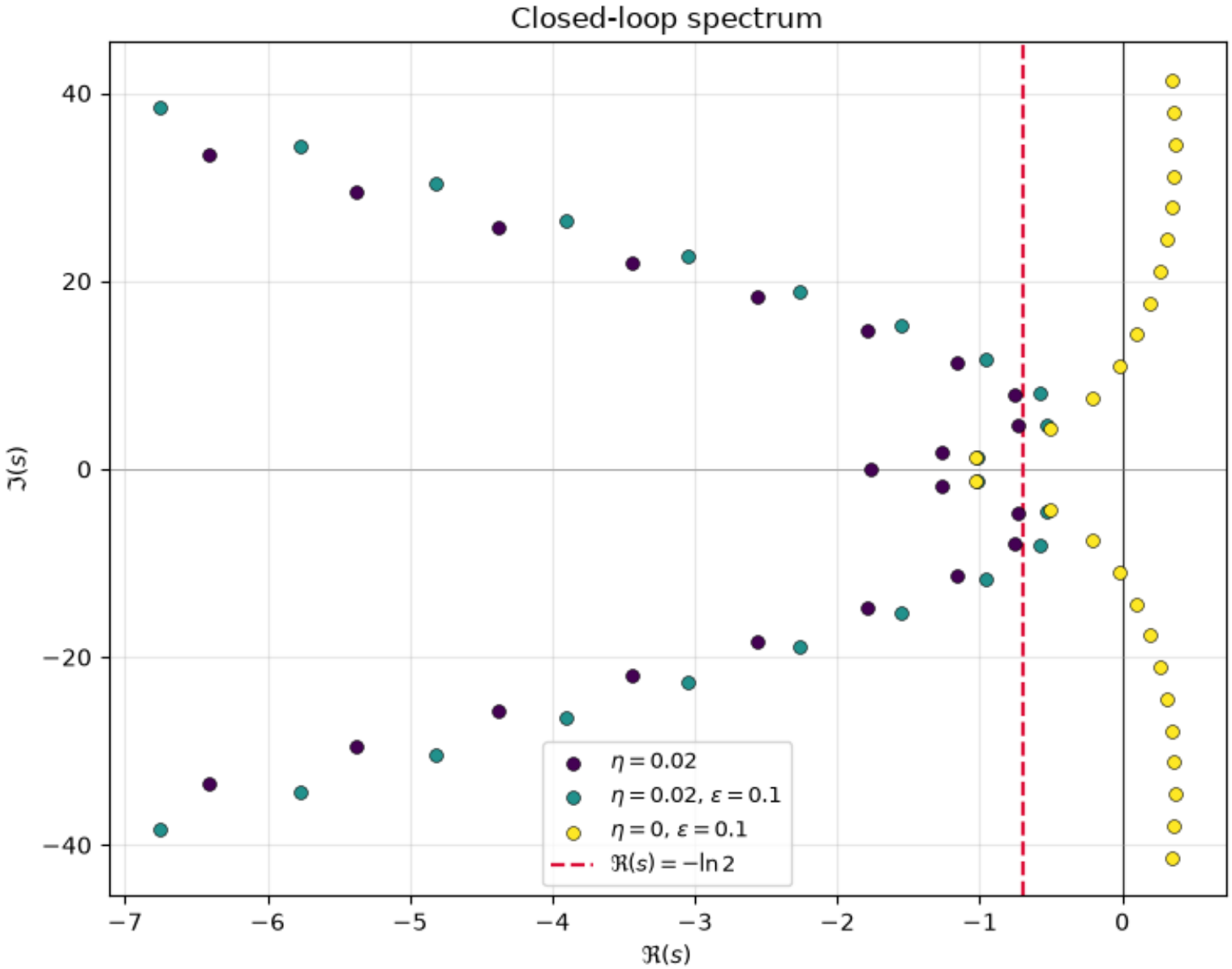
Separate in two cases: $(s_n)_{n\in\mathbb{N}}$ bounded (proved with the same contradiction as in Theorem 1) and the unbounded case.

For the unbounded case, the main idea is to prove that $f_{\eta,\varepsilon_n}(s_n) \to 0$ as $n \to \infty$. This yields

$$0 = f^2_{\eta,\varepsilon_n}(s_n) - e^{-s_n} f_{\eta,\varepsilon_n}(s_n) - 1 \to -1,$$

which is a contradiction. □

Summary: Three scenarios



Warning! Diffusion does not always stabilize

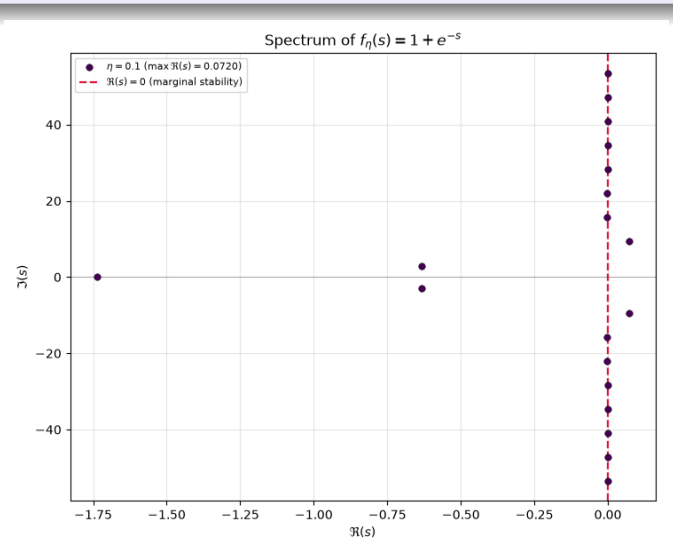
An analogous system, but with viscosity in a single channel:

$$\begin{cases} \partial_t y + \partial_x y - \eta \partial_{xx}^2 y = 0, & \partial_t \tilde{y} + \partial_x \tilde{y} = 0, \\ y(t,0) = y(t,1) - \tilde{y}(t,1), & \tilde{y}(t,0) = y(t,0), \quad \partial_x y(t,1) = 0. \end{cases}$$

Without viscosity: stable in finite time, but $\overline{\rho}(\mathbf{K}) = \sqrt{2}$ (not robust).

Conjecture (Conjecture 2)

With viscosity, for every $\varepsilon > 0$ there exists $\eta_1 > 0$ such that $\eta \in (0, \eta_1) \Rightarrow \sigma_\eta > -\varepsilon$: destabilizing effect.



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Second-order perturbation: Stability and robustness with internal coupling

$$\begin{cases} \partial_t h - \partial_x q = 0, \\ \partial_t q + c^2 \partial_x h = 0, \\ q(t,0) = U(t), \quad q(t,L) = 0, \quad Y(t) = h(t,L). \end{cases}$$



G. Bastin, J-M. Coron, A. Hayat (2025)
The usefulness of viscosity for the robustness of boundary feedback control of an unstable fluid flow system
Automatica.

Now the coupling is internal. Under Riemannian change of coordinates, we recover the previous system which we already know that is stable, but not robust under velocity perturbation.

Viscosity in the momentum equation: velocity perturbation

Diffusion is added only in the flow equation q ($c^2 = 1$):

$$\begin{cases} \partial_t h + \partial_x q = 0, \\ \partial_t q + \partial_x h - \eta \partial_{xx}^2 q = 0, \\ q(t,0) = U(t), \quad q(t,1) = 0, \quad Y(t) = h(t,1). \end{cases}$$

Now $\lambda(s)$ solves $(1+\eta s)\lambda^2 - s^2 = 0$. Similar calculations on transfer function yields

$$G(s) = \frac{2e^{\lambda(s)}}{\sqrt{1+\eta s}\,(e^{2\lambda(s)} - 1)}.$$

Closing the loop with $U(t) = -\frac{1}{2}Y(t-1)$, we obtain the characteristic equation

$$\mathcal{G}(\eta, s) = e^s \sqrt{1+\eta s}\left(e^{s/\sqrt{1+\eta s}} - e^{-s/\sqrt{1+\eta s}}\right) + 1 = 0.$$

Theorem 1: upper *and* lower bound

Theorem (Theorem 1, Bastin-Coron-Hayat 2025)

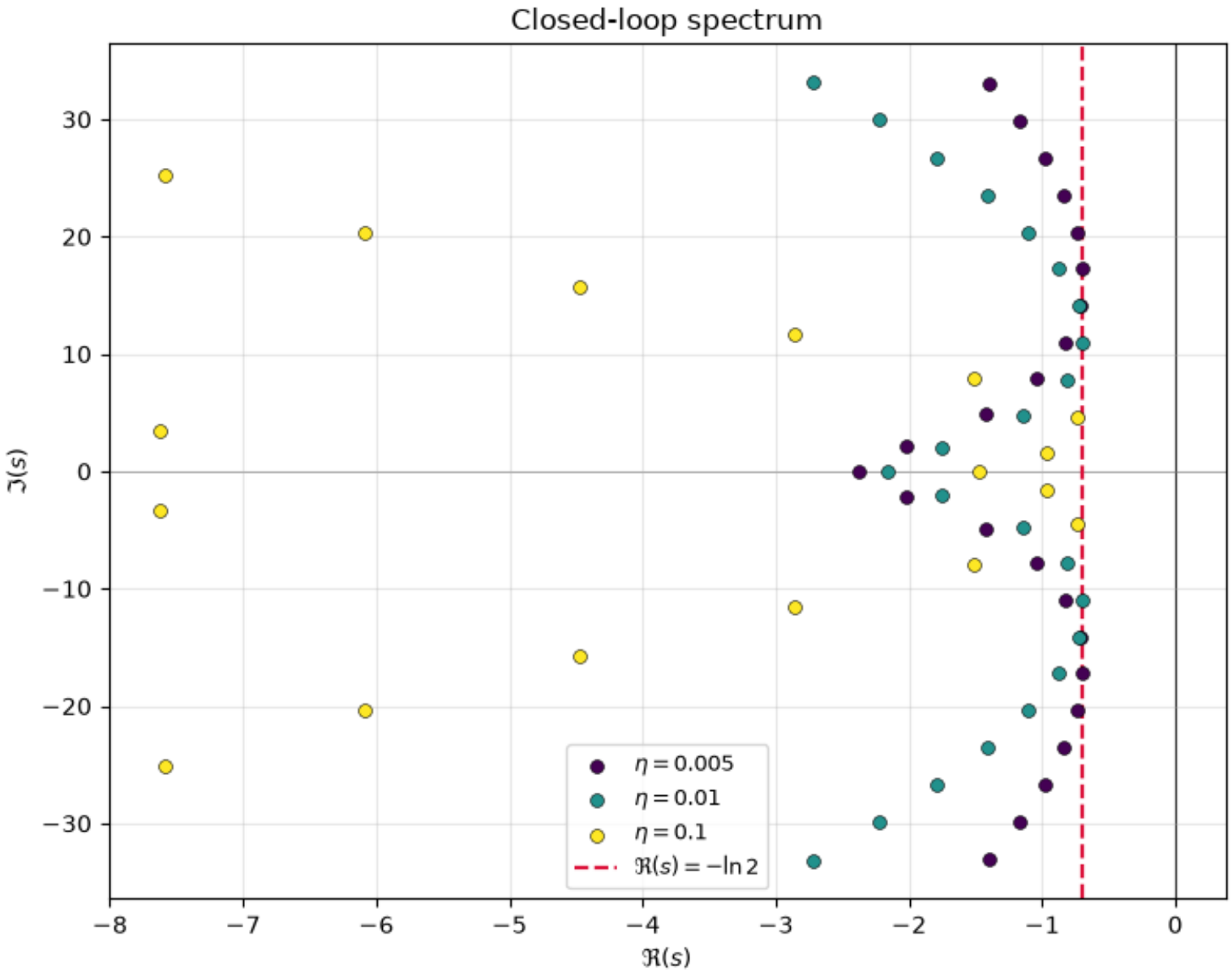
For every $\delta \in (0, \ln 2)$, *there exists* $\eta^* > 0$ *such that, for* $\eta \in (0, \eta^*]$,

$$-\ln(2) - \delta < \sigma_\eta < -\ln(2) + \delta.$$

First time the lower bound is also obtained. It solves Conjecture 1 of the previous chapter for a new system.

Key tool for the lower bound: topological degree theory. The upper bound uses the same argument as the previous chapter. We will only see the lower bound.

Closed-loop spectrum



Sketch of the proof: Structural lemmas

Lemma (Lemma 1, Bastin-Coron-Hayat, 2025)

Let $(\eta_n)_{n\in\mathbb{N}}$ be a sequence of viscosities and $(s_n)_{n\in\mathbb{N}}$ the associated sequence of poles. Denote $\sigma_n = \Re(s_n)$ and $\omega_n = \Im(s_n)$. Consider the induced sequences $(\sigma_n)_{n\in\mathbb{N}}$ and $(\eta_n\omega_n^2)_{n\in\mathbb{N}}$. Then

1. *If $\overline{\sigma}$ is an accumulation point of $(\sigma_n)_{n\in\mathbb{N}}$, then $\overline{\sigma} \in [-\infty, +\infty)$.*
2. *If $\overline{\sigma} \in (-\infty, +\infty)$ and 2θ is an accumulation point of $(\eta_n\omega_n^2)_{n\in\mathbb{N}}$, then $2\theta \in [0, +\infty)$.*

Lemma (Lemma 2, Bastin-Coron-Hayat, 2025)

Let $(\eta_n)_{n\in\mathbb{N}}$ be a sequence of viscosities and $(s_n)_{n\in\mathbb{N}}$ an associated sequence of poles, with induced sequences $(\sigma_n)_{n\in\mathbb{N}}$ and $(\eta_n\omega_n^2)_{n\in\mathbb{N}}$. Let $\overline{\sigma}$ be an accumulation point of $(\sigma_n)_{n\in\mathbb{N}}$. Then

$$\overline{\sigma} \leq -\ln(2). \tag{4}$$

Sketch of the proof: Lower bound and topological degree

Write $\sigma = \Re(s), \omega = \Im(s)$ with $\gamma^2 = \eta$ and $\varphi = \omega\gamma$. Rewrite

$$\mathcal{A}(\gamma, \sigma, \varphi) := \left(\tilde{\mathcal{G}}(\gamma, \sigma, \varphi) - \mathcal{B}(\varphi)\right) e^{-2i\varphi/\gamma}, \quad \mathcal{B}(\varphi) := -e^{-\varphi^2/2} + 1.$$

Taking $\gamma \to 0^+$ we obtain $\mathcal{A}(0, \sigma, \varphi) = e^{2\sigma + \varphi^2/2}$ and note that we have a pole for this profile when $e^{2i\varphi/\gamma} = -1$.

$$-\mathcal{A}(0, -\ln(2), \sqrt{2\ln(2)}) + \mathcal{B}(\sqrt{2\ln(2)}) = 0.$$

Idea: Find an appropiate domain where there's only 1 solution. Then connect $\mathcal{A}_\gamma$ with $\mathcal{A}_0$ by an homotopy and use

Proposition (Homotopy invariance, Coron2007, Prop. B.8)

Let $\Omega \subset \mathbb{R}^n$ be nonempty and open, $H \in C^0([0,1] \times \Omega; \mathbb{R}^n)$, and $b \in \mathbb{R}^n$ with $H(t,x) \neq b$ for all $(t,x) \in [0,1] \times \partial\Omega$. Then $\deg(H(0,\cdot), \Omega, b) = \deg(H(1,\cdot), \Omega, b)$.

A toy example

Suppose $\gamma > 0$ and let

$$H_\gamma(\sigma, \varphi) = e^{2i\varphi/\gamma} \underbrace{\left(e^\sigma + \gamma \sin(\varphi/\gamma)\right)}_{\mathcal{A}_\gamma} + 1 = 0.$$

As $\gamma \to 0^+$, we obtain $\mathcal{A}_0(\sigma, \varphi) = e^\sigma$. Hence the model map

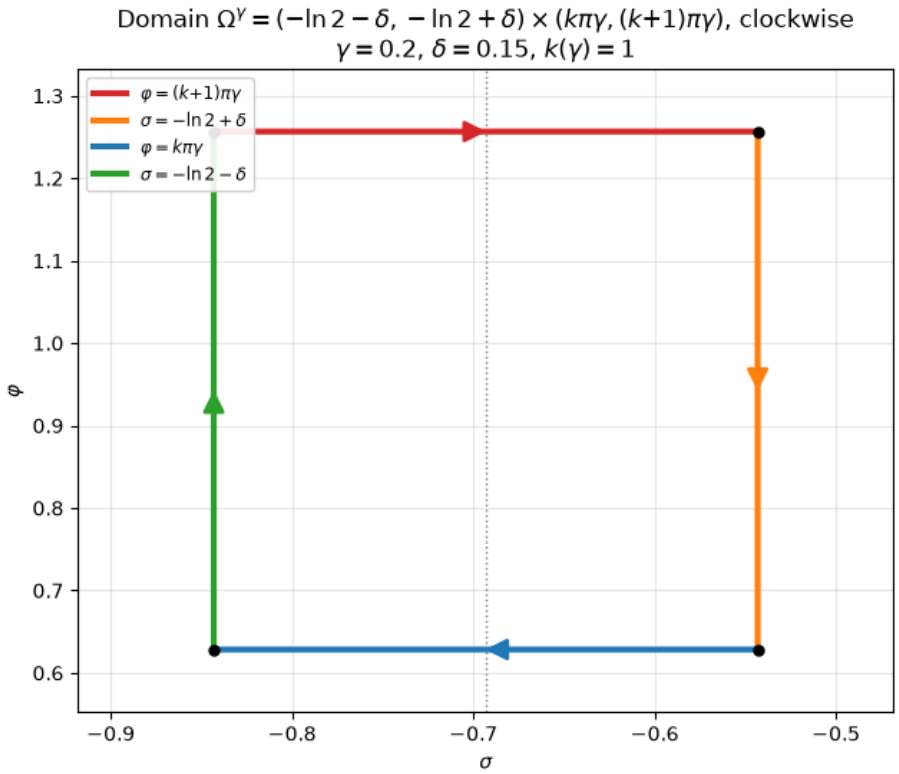
$$M_\gamma(\sigma, \varphi) = e^{2i\varphi/\gamma}(e^\sigma) + 1,$$

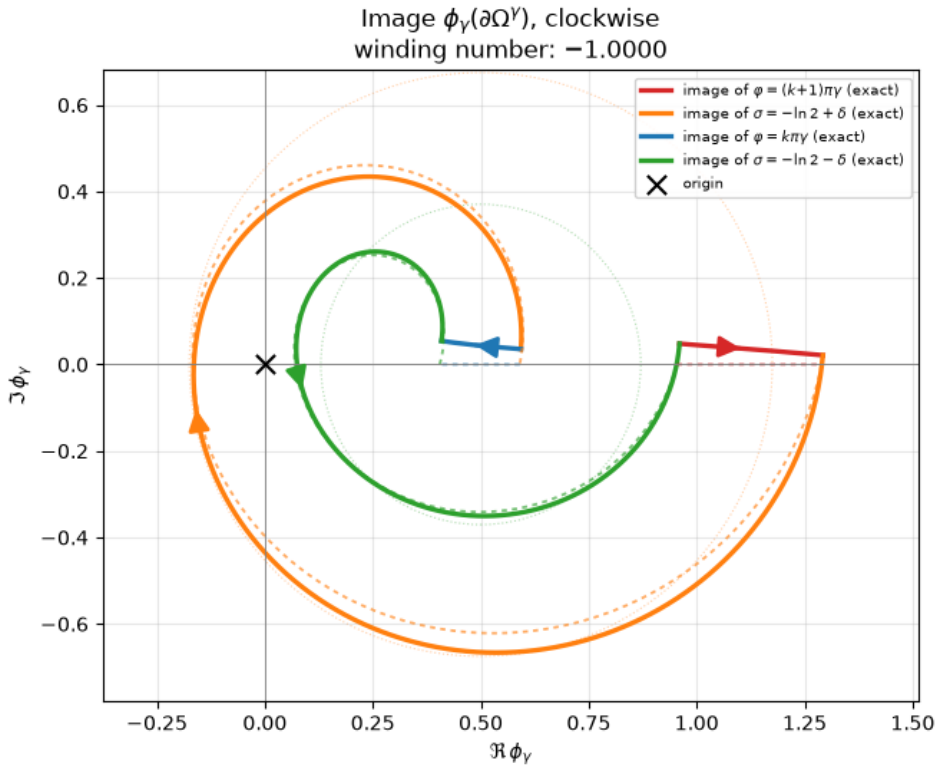
has an exact zero in the values $\sigma_* = 0$ and $\frac{2\varphi_*}{\gamma} = \pi$, where we get $\varphi_* = \frac{2\pi}{\gamma}$. Finally connect this two amplitudes by an homotopy,

$$H_\gamma^{(\mu)}(\sigma, \varphi) = e^{2i\varphi/\gamma}(\mu\mathcal{A}_\gamma + (1-\mu)\mathcal{A}_0) + 1, \quad \mu \in [0, 1],$$

obtaining, for an appropiate $\Omega_\gamma \subset \mathbb{R}^2$,

$$\deg(H_\gamma^{(1)}, \Omega_\gamma, 0) = \deg(H_\gamma^{(0)}, \Omega_\gamma, 0).$$


Domain $\Omega^\gamma = (-\ln 2 - \delta,\ -\ln 2 + \delta) \times (k\pi\gamma, (k+1)\pi\gamma)$, clockwise
$\gamma = 0.2,\ \delta = 0.15,\ k(\gamma) = 1$
$\varphi = (k+1)\pi\gamma$
$\sigma = -\ln 2 + \delta$
$\varphi = k\pi\gamma$
$\sigma = -\ln 2 - \delta$
1.3
1.2
1.1
1.0
0.9
0.8
0.7
0.6
φ
−0.9
−0.8
−0.7
−0.6
−0.5
σ


Image $\phi_\gamma(\partial\Omega^\gamma)$, clockwise
winding number: −1.0000
image of $\varphi = (k+1)\pi\gamma$ (exact)
image of $\sigma = -\ln 2 + \delta$ (exact)
image of $\varphi = k\pi\gamma$ (exact)
image of $\sigma = -\ln 2 - \delta$ (exact)
origin
0.6
0.4
0.2
0.0
−0.2
−0.4
−0.6
$\Im\,\phi_\gamma$
−0.25
0.00
0.25
0.50
0.75
1.00
1.25
1.50
$\Re\,\phi_\gamma$

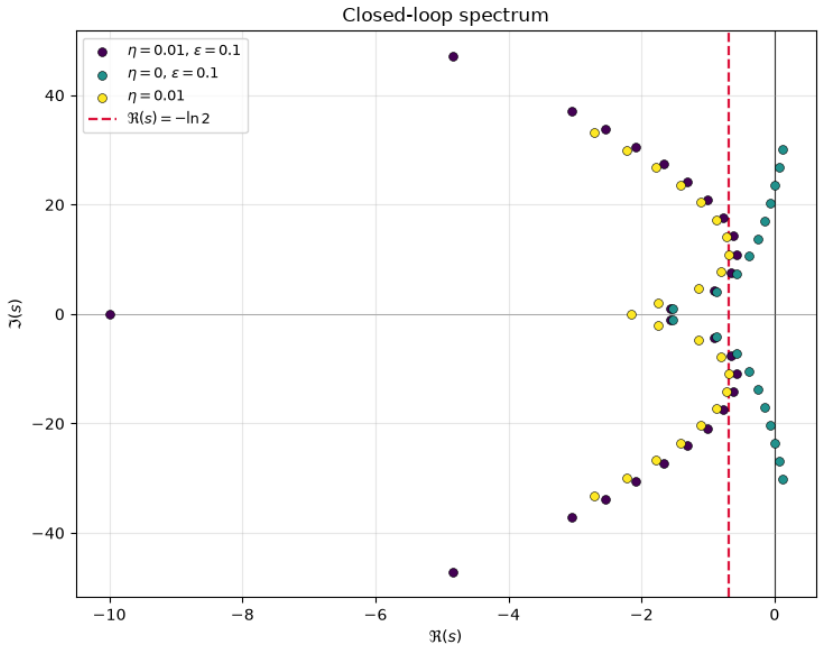
Robustness with respect to velocity

Perturbation $c^2 \mapsto 1 + \varepsilon$.

Theorem (Theorem 2, Bastin-Coron-Hayat 2025)

Let δ, η be as in Theorem 1. There exists $\varepsilon_1 > 0$ such that, for $|\varepsilon| < \varepsilon_1$,

$$\sigma_{\eta,\varepsilon} \leq -\ln(2) + 2\delta.$$



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Does the phenomenon persist at order 4?

Fourth-order dissipative terms, $\eta\partial_x^4$, arise naturally in beam and thin-film models.

Order 2: two quadratic equations $\eta\lambda^2 - \lambda - s = 0$, explicitly solvable via $z = \sqrt{1 + 4\eta s}$.

$$\begin{cases} \partial_t y_1 + \partial_x y_1 + \eta\partial_x^4 y_1 = 0, & \partial_t y_2 + \partial_x y_2 + \eta\partial_x^4 y_2 = 0, \\ y_1(t,0) = y_2(t,1) + U(t), & y_2(t,0) = y_1(t,1), \\ \partial_x^2 y_i(t,0) = \partial_x^2 y_i(t,1) = \partial_x^3 y_i(t,1) = 0, \quad i = 1,2. \end{cases}$$

NOW: The fourth-order channels give

$$\eta\lambda^4 + \lambda + s = 0,$$

a genuinely quartic equation with no analogous change of variables. This is the central obstruction of the chapter.

The system with fourth-order perturbation

$$\begin{cases} \partial_t y_1 + \partial_x y_1 + \eta \partial_x^4 y_1 = 0, & \partial_t y_2 + \partial_x y_2 + \eta \partial_x^4 y_2 = 0, \\ y_1(t,0) = y_2(t,1) + U(t), & y_2(t,0) = y_1(t,1), \\ \partial_x^2 y_i(t,0) = \partial_x^2 y_i(t,1) = \partial_x^3 y_i(t,1) = 0, \quad i = 1,2. \end{cases}$$

Each channel now needs 4 boundary conditions chosen of "no-flux" type consistent with $f_\eta(s) \to f_0(s) = e^{-s}$ as $\eta \to 0^+$.

Same plant system, just different equations!

Transfer function: no tractable closed form

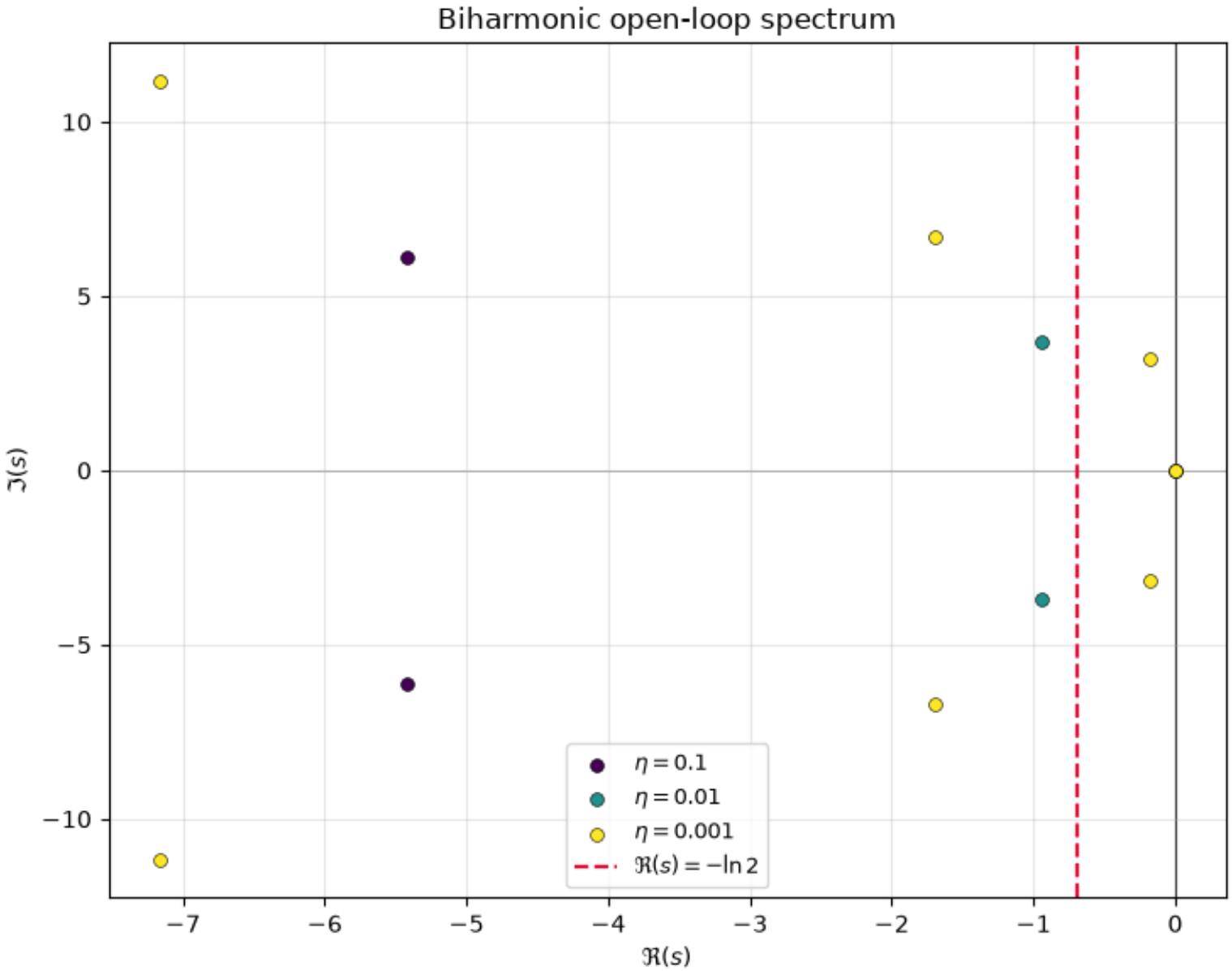
On $y_i(s,x) = \sum_{k=1}^{4} A_k(s) e^{\lambda_k(s)x}$ produces the linear system $M(s)\mathbf{A} = \mathbf{b}$, $A = (A_1, \ldots, A_4)^\top$, with

$$M(s) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 & \lambda_4^2 \\ \lambda_1^2 e^{\lambda_1} & \lambda_2^2 e^{\lambda_2} & \lambda_3^2 e^{\lambda_3} & \lambda_4^2 e^{\lambda_4} \\ \lambda_1^3 e^{\lambda_1} & \lambda_2^3 e^{\lambda_2} & \lambda_3^3 e^{\lambda_3} & \lambda_4^3 e^{\lambda_4} \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} y_i(s,0) \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

$f_\eta(s) = y_i(s,1)/y_i(s,0)$ is defined via Cramer's rule on a 4×4 system.

Is the system unstable? How to prove it?

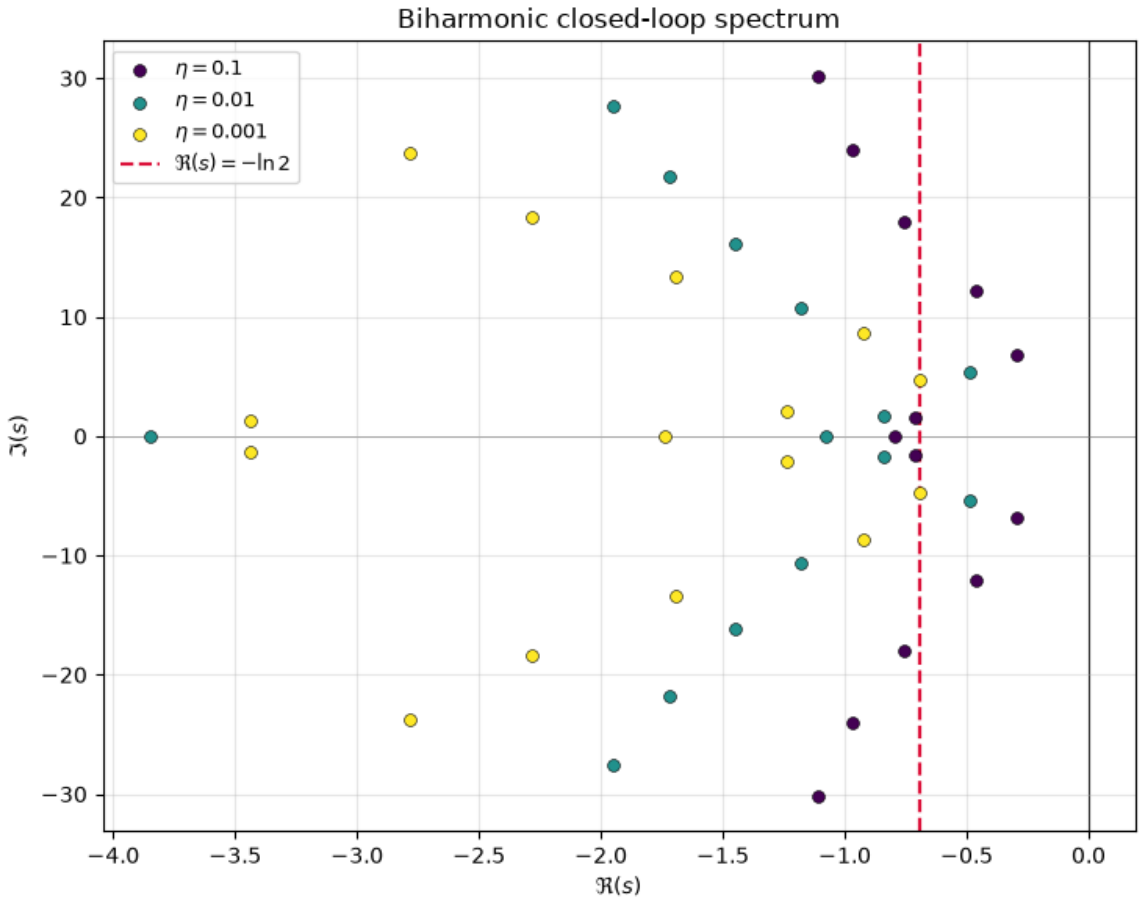
Biharmonic open-loop system: System has a pole on 0!



Closed-loop system

With the same control $U(t) = -Y(t-1)$, the characteristic equation keeps the same form as in the previous chapters,

$$f_\eta(s)^2 - f_\eta(s)e^{-s} - 1 = 0.$$



First result: Stability

Theorem (Stability)

For every $\delta > 0$, *there exists* $\eta_1 > 0$ *such that for all* $\eta \in (0, \eta_1)$,

$$\sigma_\eta \leq -\ln(2) + \delta.$$

Main ideas:

1. Asymptotic study of $f_\eta(s)$.
2. Structural lemmas for the resonant scale.
3. Once we control the error estimates, same proof as the previous two chapters!

Asymptotic classification of the roots

Proposition

As $\eta \to 0^+$, the four roots of $\eta\lambda^4 + \lambda + s = 0$ are classified into:

- *one finite branch: $\lambda_1(s) = -s + O(\eta)$ (recovers the inviscid case),*
- *three exploding branches: $\lambda_k(s) = \eta^{-1/3}\rho_k + O(1)$, $k = 2, 3, 4$, $\rho_k^3 = -1$.*

This classification replaces the lost explicit formula and allows one to expand $f_\eta(s)$.

Proposition

Let s range over a fixed compact subset of $\mathbb{C}$. Then, as $\eta \to 0^+$,

$$f_\eta(s) = e^{\lambda_1(s)} \left(1 + O(\eta^{2/3})\right), \tag{5}$$

where $\lambda_1(s)$ is the finite branch from Proposition 4.1. In particular, $f_\eta(s) \to e^{-s}$ uniformly on compacts.

f_η behavior on the resonance scale

Proposition

Let $\psi > 0$ be fixed and set

$$s = s_\eta(\sigma, \psi) := \sigma + i\eta^{-1/4}\psi, \qquad \sigma \in \mathbb{R} \text{ fixed.}$$

Then, as $\eta \to 0^+$,

$$\eta s^4 \to \psi^4, \tag{6}$$

and

$$f_\eta(s) = e^{-s-\psi^4+o(1)}, \qquad \eta \to 0^+. \tag{7}$$

In the following, consider a sequence

$$(\eta_n)_{n\in\mathbb{N}} : \ \eta_n > 0, \ \forall n \in \mathbb{N} \text{ and } \lim_{n\to+\infty} \eta_n = 0^+, \tag{8}$$

and an associated sequence of system poles

$$(s_n)_{n\in\mathbb{N}} : \quad s_n \in \mathcal{S}_{\eta_n} = \{s \in \mathbb{C} : G(s, \eta_n) = 0\}, \ \forall n \in \mathbb{N}. \tag{9}$$

Behavior of the poles

Lemma

Let $(\eta_n)_{n\in\mathbb{N}}$ be a sequence of the form (8) and $(s_n)_{n\in\mathbb{N}}$ be an associated sequence of the form (9). Then

1. $|s_n| \to +\infty$. *In other words, $(s_n)_{n\in\mathbb{N}}$ has no bounded subsequence.*
2. *If $(\sigma_n)_{n\in\mathbb{N}}$ has an accumulation point, then $|\omega_n| \to +\infty$ along that subsequence.*

The resonant scale

Lemma

Let $(\eta_n)_{n\in\mathbb{N}}$ be a sequence of the form (8) and $(s_n)_{n\in\mathbb{N}}$ be an associated sequence of the form (9). Denote $\sigma_n = \Re(s_n)$ and $\omega_n = \Im(s_n)$. Suppose $\omega_n \to +\infty$ and set

$$t_n := \eta_n \omega_n^4.$$

Let $\overline{t} \in [0, +\infty]$ be an accumulation point of $(t_n)_{n\in\mathbb{N}}$, along a subsequence such that $s_n \to \overline{s} \in \mathbb{C} \cup \{\infty\}$ appropriately. Then

1. *If $\overline{t} = 0$ or $\overline{t} = +\infty$, then $\sigma_n \to -\infty$ along the subsequence.*
2. *If $\overline{t} \in (0, +\infty)$, then*

$$\sigma_n \to g(\overline{t}) := -\frac{\overline{t}}{2} + \frac{1}{2}\ln(1 - e^{-\overline{t}}).$$

The surprising upper bound

Lemma

Let $(\eta_n)_{n\in\mathbb{N}}$ be a sequence of the form (8) and $(s_n)_{n\in\mathbb{N}}$ be an associated sequence of the form (9). Denote $\sigma_n = \Re(s_n)$ and $\omega_n = \Im(s_n)$. Then every accumulation point $\overline{s}$ of $(s_n)_{n\in\mathbb{N}}$ satisfies

$$\Re(\overline{s}) \leq -\ln(2).$$

Following the same argument as in the previous chapters, we obtain Theorem 1.

Second result: Robustness with respect to transport velocity

Perturbate $1 \mapsto 1+\varepsilon$ in the transport velocity of each channel,

$$\begin{cases} \partial_t y_1 + (1+\varepsilon)\partial_x y_1 + \eta \partial_x^4 y_1 = 0, & \partial_t y_2 + (1+\varepsilon)\partial_x y_2 + \eta \partial_x^4 y_2 = 0, \\ y_1(t,0) = y_2(t,1) + U(t), & y_2(t,0) = y_1(t,1), \\ \partial_x^2 y_i(t,0) = \partial_x^2 y_i(t,1) = \partial_x^3 y_i(t,1) = 0, & i = 1,2. \end{cases}$$

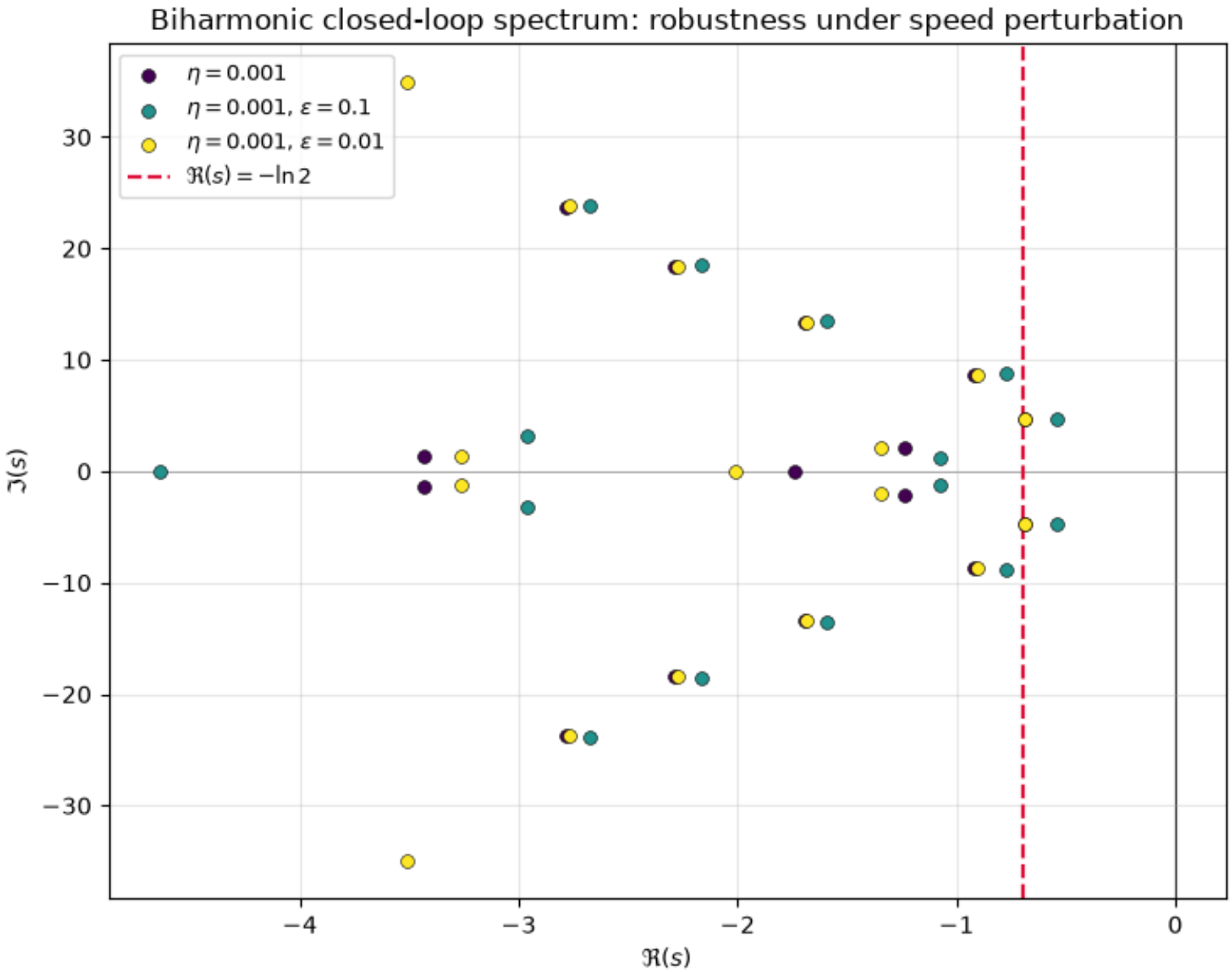
Theorem (Robustness)

Let δ, η be such that $\sigma_{\eta,0} \leq -\ln(2) + \delta$. There exists $\varepsilon_1 > 0$ such that

$$\sigma_{\eta,\varepsilon} \leq -\ln(2) + 2\delta, \qquad \forall\, \varepsilon \in (-\varepsilon_1, \varepsilon_1).$$

Sketch of the proof: The same as the previous chapters... except on the unbounded case.

Perturbated biharmonic closed-loop spectrum



Sketch of the proof: Unbounded case

Suppose by contradiction that no such ε_1 exists. Then there is a sequence $\varepsilon_n \to 0$ and $s_n \in \mathcal{S}_{\eta,\varepsilon_n}$ with

$$\Re(s_n) > -\ln(2) + 2\delta \qquad \forall n \in \mathbb{N}. \tag{10}$$

Suppose $(s_n)_{n\in\mathbb{N}}$ is unbounded and write $s_n = r_n e^{i\theta_n}$ with $\theta_n \in (-\pi, \pi]$ and $|r_n| \to +\infty$. For n large enough, $\theta_n \in (-2\pi/3, 2\pi/3)$.

Fix $\eta > 0$, set $N_n := (r_n/\eta)^{1/4} \to \infty$ and $s_n/r_n = e^{i\theta_n}$ in order to rescale to the quartic. Substituting $\lambda = N_n\xi$ into $\eta\lambda^4 + (1+\varepsilon_n)\lambda + s_n = 0$. Along any subsequence, $e^{i\theta_n} \to e^{i\alpha}$ with $\alpha \in [-2\pi/3, 2\pi/3]$ so the equation becomes

$$\xi^4 = -e^{i\alpha} \quad \Rightarrow \quad \xi_j = \exp\left[i\frac{\pi + \alpha + 2\pi j}{4}\right], \quad j = 0, 1, 2, 3.$$

Sketch of the proof: Unbounded case

This forces the existence of two branches with $\Re(\lambda_j) \to +\infty$ and two branches with $\Re(\lambda_j) \to -\infty$. Following the proof of Proposition 1, we obtain $f_{\eta,\varepsilon}(s_n) \to 0$. Also, e^{-s_n} is bounded since $|e^{-s_n}| < e^{\ln(2)-2\delta}$ for every $n \in \mathbb{N}$. This means,

$$0 = f^2_{\eta,\varepsilon_n}(s_n) - f_{\eta,\varepsilon_n}(s_n)\, e^{-s_n} - 1 \longrightarrow 0 - 0 - 1 = -1 \neq 0,$$

which is a contradiction. □

Third result: Sharpness of the constant ln(2)

Theorem (Sharpness)

For every $\delta > 0$, there exists $\eta_0 > 0$ such that for every $\eta \in (0, \eta_0)$, there exists $\tilde{s} \in \mathcal{S}_\eta$ with

$$|\Re(\tilde{s}) + \ln(2)| < \delta. \tag{11}$$

Consequently, choosing $\tilde{\eta} = \min\{\eta_0, \eta_1\}$ where η_1 comes from the Stability Theorem, we have

$$-\ln(2) - \delta \leq \sigma_\eta \leq -\ln(2) + \delta, \qquad \forall \eta \in (0, \tilde{\eta}). \tag{12}$$

Ideas on the proof:

1. Upper bound: Comes from Stability Theorem.
2. Lower bound: Topological degree. We focus on this part.

Sketch of the proof: Resonant scale and map

$$\gamma^4 = \eta, \qquad \varphi := \gamma\omega, \qquad s = \sigma + i\,\frac{\varphi}{\gamma}.$$

This means $\varphi^4 = \eta\omega^4 = t$. The critical point is

$$(\sigma_*, \varphi_*) = (-\ln(2), (\ln(2))^{1/4}).$$

Define the normalized amplitude which satisfies

$$h_\gamma(\sigma, \varphi) := e^{2s}\left(f_\eta^2(s) - f_\eta(s)e^{-s}\right) \to e^{-2\varphi^4} - e^{-\varphi^4} := h_0(\varphi) \; < \; 0$$

This gives

$$\mathcal{G}(\sigma, \varphi) = -\left(e^{-2i\varphi/\gamma}\,\mathcal{A}_\gamma(\sigma, \varphi) + \mathcal{B}\right),$$

with

$$\mathcal{A}_\gamma(\sigma, \varphi) := -e^{-2\sigma}\,h_\gamma(\sigma, \varphi), \qquad \mathcal{B} := 1. \tag{13}$$

Sketch of the proof: Remark and isomorphism

(H1) Amplitude matching at the critical point. With $t_* = \varphi_*^4 = \ln(2)$:

$$\mathcal{A}_0(\sigma_*, \varphi_*) = 4 \cdot \frac{1}{4} = 1 = \mathcal{B} \neq 0.$$

(H2) Strict monotonicity of the modulus gap. The map

$$\sigma \mapsto |\mathcal{A}_0(\sigma, \varphi_*)| - |\mathcal{B}| = e^{-2(\sigma - \sigma_*)} - 1$$

is strictly decreasing, positive for $\sigma < \sigma_*$, negative for $\sigma > \sigma_*$.

Denote by $\Psi : \mathbb{C} \to \mathbb{R}^2$ the canonical isomorphism. We will write

$$\mathcal{H}_\gamma(\sigma, \varphi) := \Psi\Big(e^{-2i\varphi/\gamma}\mathcal{A}_\gamma(\sigma, \varphi) + \mathcal{B}\Big), \quad \mathcal{A}_\gamma = P_\gamma + iQ_\gamma.$$

Hence, explicitly,

$$\mathcal{H}_\gamma = \Big(\cos\theta\, P_\gamma + \sin\theta\, Q_\gamma + 1, \quad \cos\theta\, Q_\gamma - \sin\theta\, P_\gamma\Big).$$

Sketch of the proof: Domain and model map

Fix $\delta_0 \in (0, \delta)$ and, for $\gamma > 0$ small, let

$$k(\gamma) := \left\lfloor \frac{\varphi_*}{\pi\gamma} \right\rfloor, \qquad \Omega_\gamma := [\sigma_* - \delta_0,\ \sigma_* + \delta_0] \times [k(\gamma)\pi\gamma,\ (k(\gamma)+1)\pi\gamma].$$

Define the model map where the amplitude is frozen at its $\gamma = 0$ profile, while maintaining the same phase,

$$\mathcal{M}_\gamma(\sigma, \varphi) := \Psi\Big(e^{-2i\varphi/\gamma}\, a(\sigma) + 1\Big), \qquad a(\sigma) := e^{-2(\sigma - \sigma_*)}.$$

We look for (σ, φ) such that $e^{-2i\varphi/\gamma} a(\sigma) = -1$. This yields

$$\omega = \frac{\varphi}{\gamma} = \left(k + \frac{1}{2}\right)\pi, \quad k \in \mathbb{Z}.$$

Within the band $\omega \in (k(\gamma)\pi, (k(\gamma)+1)\pi)$ there is exactly one such point, namely $\omega = \left(k(\gamma) + \frac{1}{2}\right)\pi$, giving,

$$P_* = \Big(\sigma_*,\ \left(k(\gamma) + \tfrac{1}{2}\right)\pi\gamma\Big) \in \Omega_\gamma^\circ.$$

Sketch of the proof: Degree of $\mathcal{M}_\gamma$

Write $\mathcal{M}_\gamma = (M_1, M_2)$. Hence,

$$\left.\frac{\partial M_1}{\partial \sigma}\right|_{P_*} = 2, \qquad \left.\frac{\partial M_1}{\partial \varphi}\right|_{P_*} = 0,$$

$$\left.\frac{\partial M_2}{\partial \sigma}\right|_{P_*} = 0, \qquad \left.\frac{\partial M_2}{\partial \varphi}\right|_{P_*} = \frac{2}{\gamma}.$$

Hence $\det J(P_*) = 4/\gamma > 0$, and since P_* is the unique zero of $\mathcal{M}_\gamma$ in Ω_γ, which is also non-degenerate,

$$\deg\left(\mathcal{M}_\gamma, \Omega_\gamma^\circ, 0\right) = \operatorname{sign}\left(\det J(P_*)\right) = +1.$$

Proposition (Existence of a preimage, Coron2007, Prop. B.10)

Let $(\varphi, \Omega, b) \in \mathcal{D}$ be such that $b \notin \varphi(\Omega)$. Then $\deg(\varphi, \Omega, b) = 0$.

Sketch of the proof: Homotopy and conclusion

Consider

$$\mathcal{H}_\gamma^{(\mu)}(\sigma,\varphi) := \Psi\Big(e^{-2i\varphi/\gamma}\Big[(1-\mu)\,a(\sigma)+\mu\,\mathcal{A}_\gamma(\sigma,\varphi)\Big]+1\Big), \qquad \mu\in[0,1],$$

connecting $\mathcal{H}_\gamma^{(0)} = \mathcal{M}_\gamma$ to $\mathcal{H}_\gamma^{(1)} = \mathcal{H}_\gamma$. Since $\mathcal{H}_\gamma^{(\mu)}(\sigma,\varphi)$ is non-zero on $\partial\Omega_\gamma$, by homotopy invariance of the topological degree,

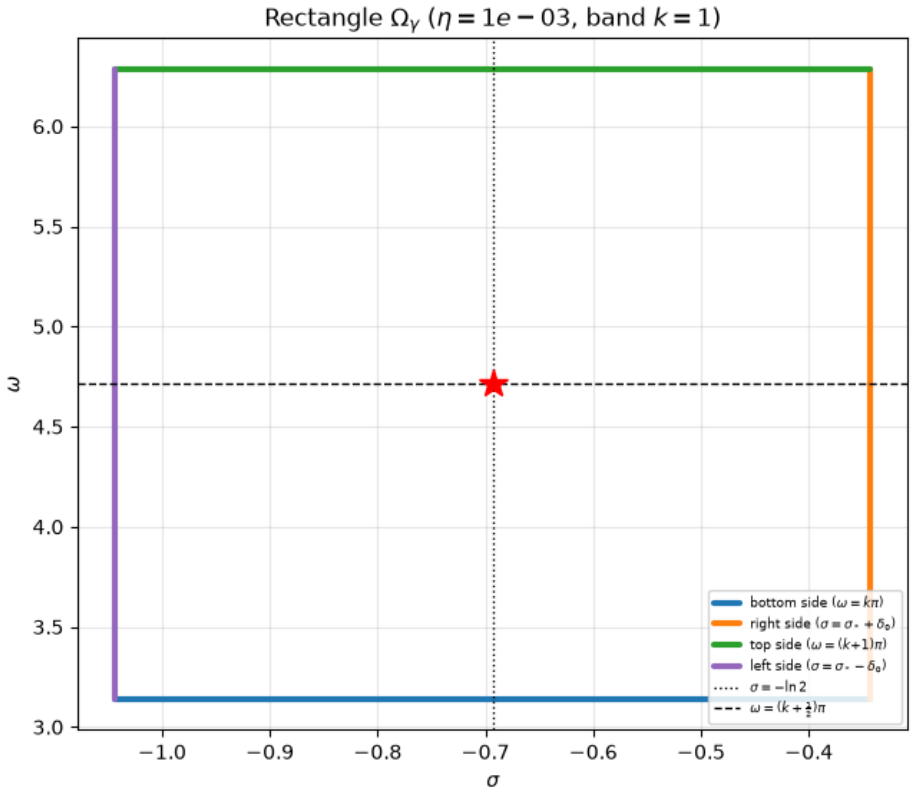
$$\deg\left(\mathcal{H}_\gamma,\Omega_\gamma^\circ,0\right) = \deg\left(\mathcal{M}_\gamma,\Omega_\gamma^\circ,0\right) = +1 \neq 0.$$

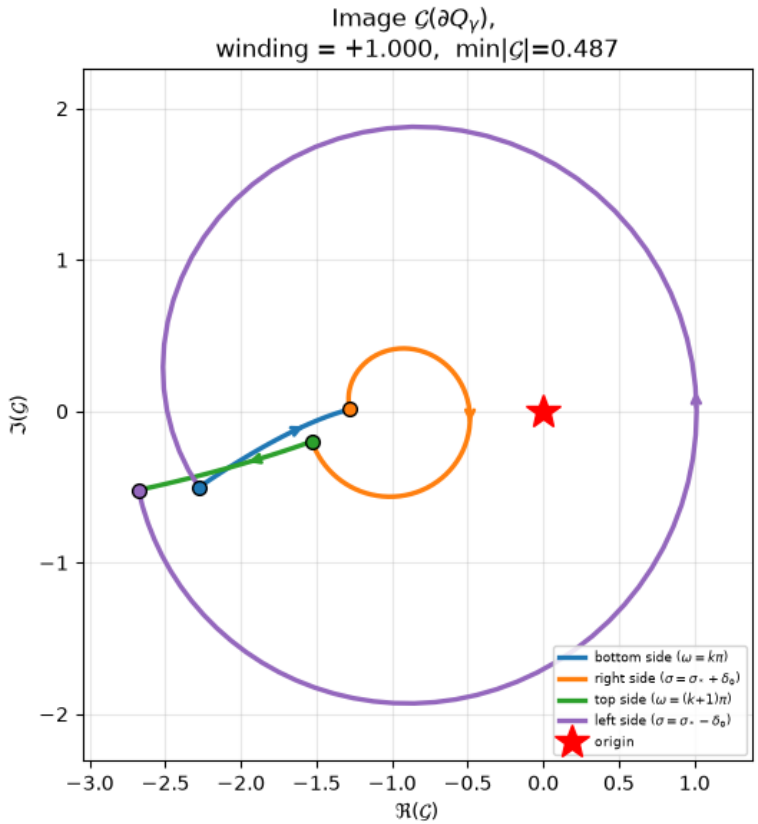
Hence we found a zero $(\tilde\sigma,\tilde\varphi)\in\Omega_\gamma$ for $\mathcal{H}_\gamma$. Since $\tilde s = \tilde\sigma + i\tilde\omega$, we get $\tilde\sigma\in(-\ln(2)-\delta_0,-\ln(2)+\delta_0)$, hence,

$$-\ln(2)-\delta_0 < \Re(\tilde s) < \sigma_\eta,$$

which ends the proof. □

The resonant rectangle and its image





Contents

1. Transfer functions: ODEs and coupled transport
2. Second-order perturbation: Stability and robustness with boundary coupling
3. Second-order perturbation: Stability and robustness with internal coupling
4. Fourth-order perturbation: Stability and robustness with boundary coupling
5. Conclusions and further work

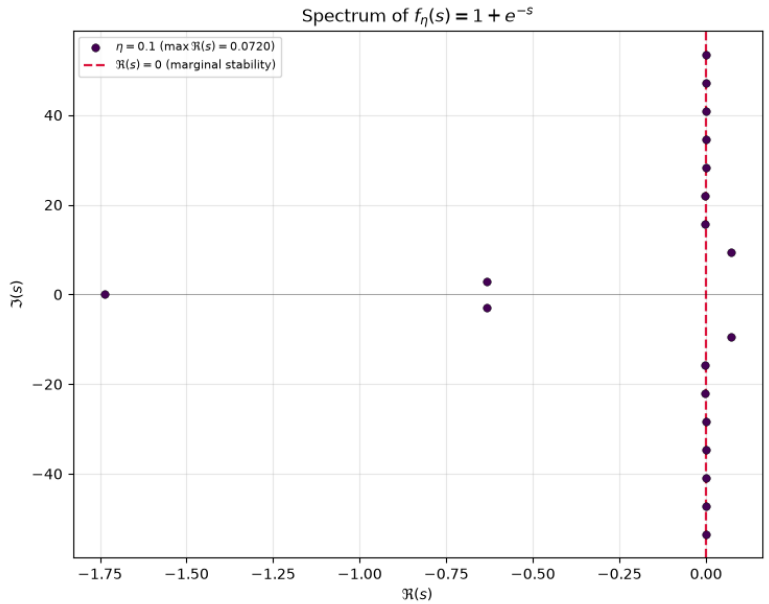
Conclusions

The mechanism small diffusion $\Rightarrow$ robustness, with optimal constant $\ln(2)$ persists under:

- Boundary coupling, order 2 (Bastin-Coron-Hayat 2023),
- Internal coupling, order 2 (Bastin-Coron-Hayat 2025),
- Boundary coupling, order 4 (Original contribution).

The simplified-system counterexample is a reminder that this is not automatic. It depends on where the dissipation acts.

$$\begin{cases} \partial_t y + \partial_x y - \eta \partial_{xx}^2 y = 0, \\ \partial_t \tilde{y} + \partial_x \tilde{y} = 0, \\ y(t,0) = y(t,1) - \tilde{y}(t,1), \\ \tilde{y}(t,0) = y(t,0), \\ \partial_x y(t,1) = 0. \end{cases}$$



Further work

1. Internal coupling, order 4?
2. Odd order perturbations.
3. Is it possible to consider $v = v(x)$?
4. Backstepping and topological degree! (To show that $\overline{\rho}(\mathbf{K}) < 1$ is a necessary condition) See



G. Bastin, J-M. Coron, A. Hayat (2025)
A necessary and sufficient robustness condition for boundary observer-controllers that stabilize 2×2 systems
Preprint on *HAL*.

Thanks for your attention!

