Stabilization of waves on exterior domains with boundary feedback

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Abstract

Suppose n is an odd number and $\mathcal{O} \subset \mathbb{R}^n$ is a bounded open set with C^∞ boundary $\partial\mathcal{O}$. Define the exterior domain $\Omega := \mathbb{R}^n \setminus \overline{\mathcal{O}}$. In this work, we want to establish an exponential stabilization result for a wave equation in a exterior domain with a pseudo-differential boundary feedback operator. The natural energy space in this case is $L^2(\Omega) \times H^{-1}(\Omega)$, where a microlocal assumption known as the exterior geometric control condition and Lax-Phillips theory are applied.

Wave equation on an exterior domain

Consider the system

$$\begin{cases} \partial_t^2 u - \Delta u = 0, \quad (0,+\infty) \times \Omega, \\ u = -\partial_\nu((-\Delta)^{-1} u_t), \quad (0,+\infty) \times \partial\Omega, \\ u(0,x) = u_0(x), \ u_t(0,x) = u_1(x). \end{cases} \tag{W_a}$$

where $(u_0, u_1) \in L^2(\Omega) \times H^{-1}(\Omega)$ and $-\Delta : H_0^1(\Omega) \to H^{-1}(\Omega)$. With this boundary feedback operator, we show that the local energy functional

$$E_R(u)(t) = \frac{1}{2}\left[\|u(t)\|^2_{L^2(\Omega_R)} + \|u_t(t)\|^2_{H^{-1}(\Omega_R)}\right], \quad t \geq 0,$$

where $\Omega_R = \Omega \cap B_R$ satisfies

$$\frac{d}{dt}E_R(u)(t) = -\int_{\partial\Omega} |\partial_\nu((-\Delta)^{-1}u_t|^2 d\sigma \leq 0.$$

Also, to establish this last property we need a microlocal assumption known as the Exterior Control Geometric Condition:

Definition 1. *Let $R > 0$ such that $\mathcal{O} \subset B_R$ and $T_R > 0$. We say that the pair $(\partial\Omega, T_R)$ satisfies the EGCC on B_R if every generalized bicharacteristic ray γ starting in $T_b^*(\mathbb{R}_t^+ \times B_R)$ at time 0 satisfies one of the following conditions:*

1. γ escapes from $\mathbb{R}_t^+ \times B_R$ before time T_R, or,

2. γ enters $\mathbb{R}_t^+ \times \omega$ at a non-diffractive point at some time between 0 and T_R.

The principal objective is to prove the next result.

Theorem 2. *Let $R, T_R > 0$ such that $(\partial\Omega, T_R)$ satisfies EGCC in B_R. Then, there exists $C > 0$ and $\alpha > 0$ such that for every $(f_1, f_2) \in L^2(\Omega) \times H^{-1}(\Omega)$ supported on B_R, the solution u of (W_a) with initial data (f_1, f_2) verifies*

$$E_R(u)(t) \leq Ce^{-\alpha t} E_R(u)(0).$$

This problem was studied by Aloui and Khenissi in 2002 for different feedback mechanisms: internal damping in one case, and Neumann-type boundary feedback in the other.

Decay rate for $U_a(t)$

Writing the system (W_a) as a first-order system we obtain

$$\partial_t \begin{pmatrix} u \\ v \end{pmatrix} = A_a \begin{pmatrix} u \\ v \end{pmatrix}, \qquad A_a = \begin{pmatrix} 0 & I \\ \Delta & 0 \end{pmatrix},$$

with domain

$$D(A_a) = \{(y_1, y_2) \in H : (y_2, \Delta y_1) \in H, \ y_1 + \partial_\nu(Gu_t) = 0 \text{ on } \partial\Omega\}.$$

With this domain we can prove

Proposition 3. *A_a is a maximal dissipative operator.*

This together with Hille-Yosida Theorem proves that $(U_a(t))_{t\geq 0}$ is a contraction semigroup.

Bicharacteristics and Hamiltonian flow

Definition 4. *We define bicharacteristic curve as the maximal solution of the Hamiltonian system*

$$\begin{cases} \dot{x} = \frac{\partial p}{\partial \xi}, \\ \dot{\xi} = -\frac{\partial p}{\partial x}, \end{cases}$$

which we denote by $\gamma : s \in I \to (x(s), \xi(s))$ where $I \subseteq \mathbb{R}$ is a maximal interval. The null-bicharacteristics are the curves that satisfy

$$\frac{d}{ds}\gamma(s) = H_p(\gamma(s)), \quad H_p(f) = \{p, f\} = \sum_{j=1}^n \left(\frac{\partial p}{\partial \xi_j}\frac{\partial f}{\partial x_j} - \frac{\partial p}{\partial x_j}\frac{\partial f}{\partial \xi_j}\right),$$

where H_p denotes the Hamiltonian flow.

Lax-Phillips theory

To study the local energy, we introduce the Lax-Phillips semigroup associated to the system (W_a).

Definition 5. *We define the spaces of outgoing and incoming data for undamped system as*

$$D_+ = \{(\varphi_1, \varphi_2) \in H : U_a(t)\varphi = 0, \text{on } \ |x| < t + R, \ t \geq 0\},$$
$$D_- = \{(\varphi_1, \varphi_2) \in H : U_a(t)\varphi = 0, \text{on } \ |x| < -t + R, \ t \leq 0\}.$$

For $t \geq 0$ we define the Lax-Phillips semigroup as

$$Z_a(t) = P^+ U_a(t) P^-,$$

where $P^\pm$ is the projection onto $D_\pm^\perp$.

Some good properties about this semogroup are

Proposition 6. *1. $Z_a(t)D_+ = Z_a(t)D_- = \{0\}$.*

2. $Z_a(t)$ operates over $K = D_+^\perp \cap D_-^\perp$.

3. $(Z_a(t))_{\geq 0}$ forms a semigroup over K.

Microlocal defect measures

This notion was introduced by Gérard in 91' and Tartar in 90'. The main idea is to study the lack of compactness of the operator.

Definition 7. *Let $\{u_n\}_{n\in\mathbb{N}} \subseteq L^2_{loc}(\Omega)$ be a bounded sequence which weakly converges to 0. Then there exists a subsequence $\{u_{n_k}\}_{k\in\mathbb{N}}$ and a positive Radon measure μ over $\Omega \times S^{n-1}$ such that every essentially homogeneous pseudo-differential operator $A \in \Psi_c^0(\Omega)$ with principal symbol $\sigma_0(A)$ satisfies*

$$(A(\chi u_{n_k}), \chi u_{n_k})_{L^2(\Omega)} \to \int_{\Omega \times S^{n-1}} \sigma_0(A)(x,\xi) d\mu,$$

for every $\chi \in C_0^\infty(\Omega)$ such that $\chi = 1$ in $\pi_x(\text{supp}(\sigma_0(A)))$. We'll say that μ is the microlocal defect measure attached to the sequence $\{u_n\}_{n\in\mathbb{N}}$.

The main properties about MDM are the following.

Theorem 8. *Let P be a differential operator of order m over Ω and $\{u_n\}_{n\in\mathbb{N}} \subseteq L^2_{loc}(\Omega)$ be a bounded sequence that converges weakly to 0 with MDM μ. Then the following assertions are equivalent.*

1. $Pu_n \to 0$ strong in $H^{-m}_{loc}(\Omega)$, where $m > 0$,

2. $\text{supp}(\mu) \subset \{(x,\xi) \in \Omega \times S^{n-1} : \sigma_m(P)(x,\xi) = 0\}$.

Theorem 9. *Let P be a differential operator of order m over Ω and $\{u_n\}_{n\in\mathbb{N}} \subseteq L^2_{loc}(\Omega)$ be a bounded sequence that converges weakly to 0 with MDM μ. Suppose that $Pu_n \to 0$ in $H^{-(m-1)}_{loc}(\Omega)$. Then $\text{supp}(\mu)$ is a union of curves $s \in I \mapsto \left(x(s), \frac{\xi(s)}{|\xi(s)|}\right)$ where $s \in I \mapsto (x(s), \xi(s))$ is a null-bicharacteristic of p, the principal symbol of P.*

Comments about the theorem and previous results

The main idea of the proof is to prove that $\mu = 0$ in a subset of the incoming and outgoing data which means that the wave operator is compact in that subset. Due to the previous theorem, we can propagate the compactness using the null-bicharacteristics, where the EGGC is used. This is the scheme of the work of Aloui and Khenissi 2002.

There are many difficulties in this part since we're dealing with a problem where the bicharacteristic interacts with the boundary. The construction of MDM in manifolds with boundary can be consulted in the work of Lebeau 92'. Using a Neumann condition, Aloui managed to adapt the argument to this case.

Another difference is that we have to work in the compressed cotangent bundle of Melrose-Sjöstrand denoted by T_b^*M where M is a manifold with boundary. The idea behind this space is to glue the boundary and the interior of the manifold in a smooth way so we can recover information from the boundary and propagate it to the interior using a Lifting Lemma, first introduced by Bardos, Lebeau and Rauch in 92. We also have a new notion of generalized bicharacteristic rays which studies the possible interaction of the bicharacteristics with the boundary.