Feedback Stabilization of the Linear Kuramoto-Sivashinsky equation with Saturated Controls

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11 de Diciembre de 2024

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About the equation

The Kuramoto-Sivashinsky equation is a non linear parabolic equation. In this presentation we will consider the 1-dimensional equation

$$w_t + w_{xxxx} + \lambda w_{xx} + w w_x = 0,$$

where $\lambda > 0$ is the anti-diffusion parameter. The equation was developed independently by

- Kuramoto on 1975 as a model for reaction-diffusion phenomenoms
- Sivashinsky on 1977 as a model for laminar flame front.

Some results



W. Liu, and M. Krstić (2001)
Stability enhancement by boundary control in the Kuramoto–Sivashinsky equation.

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E. Cerpa (2010)
Null controllability and stabilization of the linear Kuramoto-Sivashinsky equation.

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Spectral analysis

Given the equation

$$w_t + w_{xxxx} + \lambda w_{xx} = 0,$$

consider the spacial operator associated given by

$$\begin{cases} A : \mathcal{D}(A) \subset L^2(0,L) \to L^2(0,L), \\ \mathcal{D}(A) = H^4(0,L) \cap H_0^2(0,L), \\ Aw = -w'''' - \lambda w''. \end{cases} \tag{1}$$

A is self-adjoint with compact resolvent. The spectrum of A is a discrete decreasing subset $\{\sigma_j\}_{j\in\mathbb{N}}$ of $\mathbb{R}$ satisfying $\lim_{j\to\infty} \sigma_j = -\infty$.

The eigenfunctions $\{e_j\}_{j\in\mathbb{N}}$ form an orthonormal basis of $L^2(0,L)$:

$$\begin{cases} -e_j'''' - \lambda e_j'' = \sigma_j e_j \\ e_j(0) = e_j(L) = e_j'(0) = e_j'(L) = 0. \end{cases} \tag{2}$$

Spectral analysis

Moreover, we have that

$$\sigma_j \leq \frac{\lambda^2}{4}, \quad \forall j \in \mathbb{N}.$$



E. Cerpa (2010)
Null controllability and stabilization of the linear Kuramoto-Sivashinsky equation.

Since $\lim_{j\to\infty} \sigma_j = -\infty$, there are, at most, finite non-negative eigenvalues.

Idea: Try to move those eigenvalues that don't sound as we want.

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Scalar control problem

We will consider the system

$$\begin{cases} w_t + w_{xxxx} + \lambda w_{xx} = u, & t > 0,\ x \in (0, L), \\ w(t, 0) = w(t, L) = 0, & t > 0, \\ w_x(t, 0) = w_x(t, L) = 0, & t > 0, \\ w(0, x) = w_0(x), & x \in (0, L). \end{cases} \tag{3}$$

where $w_0 \in L^2(0, L)$. We want to find a control u such that

$$||w||_{L^2(0,L)} \leq e^{-\omega t} ||w_0||_{L^2(0,L)}, \qquad \forall t \geq 0,$$

for $\omega > 0$. This result is known as exponential stabilization of the $L^2(0, L)$ norm.

Consequences of spectral analysis

Let $\omega > 0$ be the desired decay rate, and let $n \in \mathbb{N}$ be the number of non-negative and slow enough eigenvalues such that

$$-\sigma_j > \omega > 0, \qquad \forall j \geq n+1.$$

Idea: Find a control $u(t,x)$ written as

$$u(t,x) = \sum_{j=1}^{n} G_j(t) e_j(x)$$

which affects only the first n eigenvalues. We want to find the control gains $G_j(t)$.



P. Guzmán, and C. Prieur (2020)
Rapid stabilization of a reaction-diffusion with distributed disturbance.
.

Expansion of the solution

Due to the spectral analysis, we can expand solutions of the equation as

$$w(t,x) = \sum_{j=1}^{\infty} w_j(t) e_j(x). \tag{4}$$

and expand the $L^2(0,L)$ norm as

$$E(t) = \frac{1}{2}\int_0^L |w|^2 dx = \frac{1}{2}\sum_{j=1}^{\infty} w_j^2(t).$$

the objective is to prove

$$E(t) \leq e^{-2\omega t} E(0), \qquad \forall t \geq 0.$$

Construction of the control

Multiplying (3) by w and integrating in $(0, L)$, we get that

$$\frac{1}{2}\frac{d}{dt}\left(\int_0^L |w|^2 dx\right) + \int_0^L -w(u - w_{xxxx} - \lambda w_{xx})dx = 0. \tag{5}$$

Using the spectral analysis again, we find that

$$\begin{aligned}\int_0^L -w(u - w_{xxxx} - \lambda w_{xx})dx &= \int_0^L -w(A + u)dx \\ &= \sum_{j=1}^{\infty} -\sigma_j w_j^2(t) + \sum_{j=1}^{n} -w_j(t)G_j(t).\end{aligned}$$

Here we used that $\langle e_i, e_j \rangle_{L^2(0,L)} = \delta_{ij}$

Exponential stability

In order to use the upper bound of ω, we note that

$$\int_0^L -w(A+u)dx = \sum_{j=n+1}^{\infty} -\sigma_j w_j^2(t) + \sum_{j=1}^{n} \left(-\sigma_j w_j^2(t) - w_j(t)G_j(t)\right)$$

For the blue part, we use that $-\sigma_j > \omega > 0,\ \forall j \geq n+1$ to obtain

$$\sum_{j=n+1}^{\infty} -\sigma_j w_j^2(t) > \omega \sum_{j=n+1}^{\infty} w_j^2(t). \tag{6}$$

Exponential stability

For the red part, we need that

$$G_j(t) = -(\omega + \sigma_j) w_j(t), \qquad \forall j \in \{1, ..., n\}. \tag{7}$$

In this case, we find that

$$\sum_{j=1}^{n} \left(-\sigma_j w_j^2(t) - w_j(t) G_j(t)\right) = \omega \sum_{j=1}^{n} w_j^2(t).$$

Therefore, using the two previous estimations we find out that under the control

$$u(t,x) = \sum_{j=1}^{n} (-\omega - \sigma_j) w_j(t) e_j(x), \tag{8}$$

Exponential stability

we get the estimation

$$\int_0^L -w(A+u)dx \geq \omega \sum_{j=n+1}^{\infty} w_j^2(t) + \omega \sum_{j=1}^{n} w_j^2(t) = \omega \sum_{j=1}^{\infty} w_j^2(t) = 2\omega E(t).$$

Hence, we get that

$$0 \geq \dot{E}(t) + 2\omega E(t)$$

where we deduce

$$E(t) \leq e^{-2\omega t} E(0), \qquad \forall t \geq 0,$$

or equivalently

$$||w||_{L^2(0,L)} \leq e^{-\omega t} ||w_0||_{L^2(0,L)}, \qquad \forall t \geq 0.$$

Principal result

Theorem

Let w be a solution of (3) with initial value $w_0 \in L^2(0, L)$ and a decay rate $\omega > 0$. Then there exists a closed feedback law

$$u(t, x) = \sum_{j=1}^{n} (-\omega - \sigma_j) w_j(t) e_j(x), \tag{9}$$

such that (3) is exponentially stable in $L^2(0, L)$ with decay rate ω. Equivalently, the following inequality holds

$$||w(t, \cdot)||_{L^2(0,L)} \leq e^{-\omega t} ||w_0||_{L^2(0,L)}, \qquad t \geq 0, \tag{10}$$

where $\{\sigma_j\}_{j \in \mathbb{N}}$ and $\{e_j\}_{j \in \mathbb{N}}$ denote the eigenvalues and eigenfunctions given by the spectral analysis.

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Saturated controls

We define the saturation of $u \in \mathbb{R}^m$ as

$$\mathfrak{sat}(u)_k = \begin{cases} u_k, & |u_k| \leq \ell, \\ \ell \frac{u_k}{|u_k|}, & |u_k| \geq \ell, \end{cases}$$

where $\ell > 0$ is the level of saturation.



A. Mironchenko, C. Prieur and F. Wirth (2020)
Local stabilization of an unstable parabolic equation via saturated controls.

Problem formulation

We will study the problem

$$\begin{cases} w_t + w_{xxxx} + \lambda w_{xx} = v, & t > 0,\ x \in (0, L), \\ w(t, 0) = w(t, L) = 0, & t > 0, \\ w_x(t, 0) = w_x(t, L) = 0, & t > 0, \\ w(0, x) = w_0(x), & x \in (0, L). \end{cases} \tag{11}$$

where

$$v = \sum_{k=1}^{m} b_k(x) \mathfrak{sat}(u_k(t)),$$

and spectral analysis still applies!

Problem formulation

Given a control $u : (0, +\infty) \to \mathbb{R}^m$, the puntual saturation is

$$\mathfrak{sat}(u(t)) = \begin{bmatrix} \mathfrak{sat}(u_1(t)) \\ \vdots \\ \mathfrak{sat}(u_m(t)) \end{bmatrix},$$

and consider

$$\mathbf{b}(x) = (b_1, ..., b_m)$$

where $b_k : [0, L] \to \mathbb{R}$ are the localizations in $L^2(0, L)$ of the u_k component. Following the previous steps, we find

$$\dot{w}_j(t) = \sigma_j w_j(t) + \mathbf{b}_j \cdot \mathfrak{sat}(u(t)), \qquad \forall j \in \mathbb{N}, \tag{12}$$

where $\mathbf{b}_j = \langle e_j, \mathbf{b}(x) \rangle_{L^2(0,L)} = (b_{j1}, ..., b_{jm}) \in \mathbb{R}^{1\times m}$ for every $j \in \mathbb{N}$.

Problem formulation

From the spectral analysis, we take $\eta > 0$ such that

$$\begin{cases} \sigma_j < -\eta < 0, & \forall j \geq n+1, \\ \sigma_j \geq 0, & \forall j \leq n. \end{cases} \tag{13}$$

Where $n \in \mathbb{N}$ denote the non-negative eigenvalues. Therefore, using the notation

$$z := \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix}, \quad \mathbf{A} := \begin{bmatrix} \sigma_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \sigma_n \end{bmatrix}, \quad \mathbf{B} := \begin{bmatrix} b_{11} & \cdots & b_{1m} \\ \vdots & \ddots & \vdots \\ b_{n1} & \cdots & b_{nm} \end{bmatrix}, \tag{14}$$

we find the (non-linear) finite dimensional system

$$\dot{z}(t) = \mathbf{A}z(t) + \mathbf{B}\mathfrak{sat}(u(t)). \tag{15}$$

Functional spaces for the control

Let X_n be the subspace of $L^2(0,L)$ generated by $\{e_j\}_{j=1}^n$. Let $\pi_n : L^2(0,L) \to X_n$ be the orthogonal projection

$$\pi_n(w(t,\cdot)) = \sum_{k=1}^{n} w_k(t) e_k(\cdot),$$

then we have the decomposition

$$L^2(0,L) = X_n \oplus X_n^{\perp},$$

where $X_n^{\perp}$ denotes the orthogonal complement of X_n.

Functional spaces for the control

Using Parseval's identity, the decomposition is isomorphic to

$$\ell^2(\mathbb{R}) = \mathbb{R}^n \oplus \ell^2_{j>n}(\mathbb{R}),$$

where

$$\ell^2_{j>n}(\mathbb{R}) = \left\{ \{x_k\}_{k\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}} : \quad x_k = 0, \forall k \in \{1, ..., n\} \wedge \sum_{k=n+1}^{\infty} |x_k|^2 < \infty \right\}.$$

Feedback law

Take $\mathbf{K} : X_n \to \mathbb{R}^m$ and consider the control

$$u(t) = \mathbf{K}z(t) \tag{16}$$

then we have the system

$$\dot{w}_j(t) = \sigma_j w_j(t) + \mathbf{b}_j \cdot \mathfrak{sat}(\mathbf{K}z(t)), \qquad j \in \mathbb{N}, \tag{17}$$

where the first $n \in \mathbb{N}$ equations can be written as

$$\dot{z}(t) = \mathbf{A}z(t) + \mathbf{B}\mathfrak{sat}(\mathbf{K}z(t)). \tag{18}$$

From (18) and (17) we see that saturation induces a coupling between the finite unstable system and the infinite stable system

Cascade system

$$\dot{z}(t) = \mathbf{A}z(t) + \mathbf{B}\mathfrak{sat}(\mathbf{K}z(t))$$

$$z(t)$$

$$\dot{w}_j(t) = \sigma_j w(t) + \mathbf{b}_j \cdot \mathfrak{sat}(\mathbf{K}z(t)), \quad j \geq n+1$$

$$w_j(t)$$

$$\dot{w}_j(t) = \sigma_j w(t) + \mathbf{b}_j \cdot \mathfrak{sat}(\mathbf{K}z(t)), \quad j \in \mathbb{N}$$

Figure: Cascade system justification

Region of attraction

As a consequence of saturations, we have regions of attraction:

Definition

$S \subset \mathbb{R}^n$ is a region of attraction if

1. $0 \in \operatorname{int} S$;
2. *For every initial condition $z_0 \in S$ we have that $z(t) \to 0$ if $t \to \infty$;*
3. *For every initial condition $z_0 \in S$, it follows that $z(t) \in S$ for all $t \geq 0$.*

Region of attraction and stabilization

Proposition

Choose $\mathbf{K} : X_n \to \mathbb{R}^m$ *such that the finite dimensional system is locally exponentially stable in* $0 \in \mathbb{R}^n$ *with region of attraction* $S \in \mathbb{R}^n$. *Then:*

1. *System* (17) *is locally exponentially stable in 0 with region of attraction* $S \times \ell^2_{j>n}(\mathbb{R})$.
2. *System* (11) *is locally exponentially stable in 0 with region of attraction* $i(S) \times X_n^{\perp}$.

In addition, for any closed and bounded set $G \subseteq \text{int}(i(S) \times X_n^{\perp})$, *there exist two positive values* M *and* a *such that for any initial condition* $w(0,\cdot) \in G$, *the solution* $w(\cdot)$ *satisfies*

$$||w(t,\cdot)||_{L^2(0,L)} \leq Me^{-at} ||w(0,\cdot)||_{L^2(0,L)}, \quad t \geq 0.$$

Idea behind the proposition

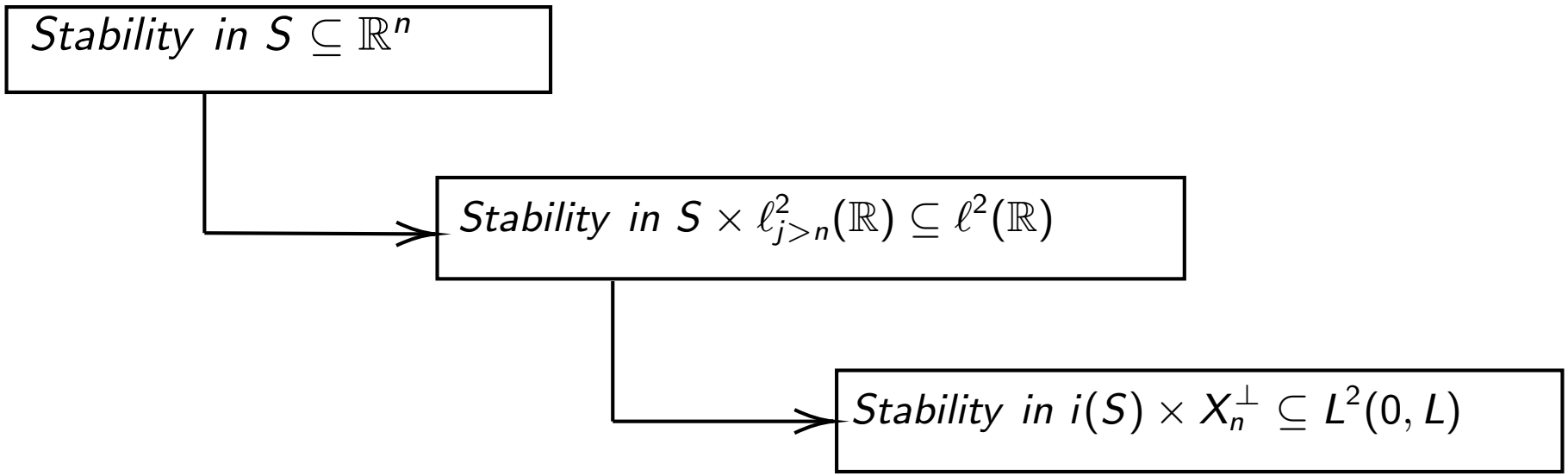


Figure: Relationship with the cascade system

Sketch of the proof

Suppose that we have

$$|z(t)| \leq Me^{-at}|z_0|, \qquad t \geq 0. \tag{19}$$

Take $(w_{n+1}(0), w_{n+2}(0), ...) \in \ell^2_{j>n}(\mathbb{R})$, then by Duhamel's formula we have

$$w_j(t) = e^{\sigma_j t} w_j(0) + \mathbf{b}_j \cdot \int_0^t e^{\sigma_j (t-s)} \mathfrak{sat}(\mathbf{K}z(s)) ds, \qquad \forall j \geq n+1.$$

Using $\sigma_j < -\eta$ for $j \geq n+1$ and the estimate

$$|\mathfrak{sat}(\mathbf{K}z)| \leq |\mathbf{K}z| \leq ||\mathbf{K}||\ |z|.$$

we achieve

Sketch of the proof

$$|w_j(t)| \leq e^{-\eta t}|w_j(0)| + M|\mathbf{b}_j|\,||\mathbf{K}||\,|z_0|\left(\frac{e^{-at}-e^{-\eta t}}{\eta - a}\right).$$

By Bessel's inequality we get

$$\sum_{j=n+1}^{\infty} |\mathbf{b}_j|^2 \leq \sum_{k=1}^{m} ||b_k||^2_{L^2(0,L)}\,,$$

therefore using Young's inequality and summing over $j \in \mathbb{N}$ we get

$$\sum_{j\in\mathbb{N}} |w_j(t)|^2 \leq Ce^{-2at} \sum_{j\in\mathbb{N}} |w_j(0)|^2$$

where

$$C = \max\left\{2, \frac{2M^2\,||\mathbf{K}||^2 \sum_{k=1}^{m} ||b_k||^2_{L^2(0,L)}}{(\eta - a)^2}\right\} > 0.$$

Meaning of the proposition

The last inequality give us the local exponential stability in $\ell^2(\mathbb{R})$. Via Parseval's identity, the result in $L^2(0, L)$ follows.

This proposition says that if we prove that the unstable finite dimensional system is stable, the stability of the whole system follows.

Problem: Even if the pair $(\mathbf{A}, \mathbf{B})$ is Hurwitz, our finite dimensional could be unstable!

Idea: Decompose the saturated control in a linear and a geometric component.

On the stability of the pair $(\mathbf{A}, \mathbf{B})$

We'll see that the condition of controllability of $(\mathbf{A}, \mathbf{B})$ is feasible, where $m \in \mathbb{N}$ is the number of columns of $\mathbf{B}$.

1. If $m = 1$, by Kalman's condition and Vandermonde's determinant we get

$$\det\left(\mathbf{B}, \mathbf{AB}, \cdots, \mathbf{A}^{n-1}\mathbf{B}\right) = \prod_{j=1}^{n} b_j \prod_{0 \leq i < k \leq n} (\sigma_k - \sigma_i),$$

   which means that if $b_j \neq 0$ and eigenvalues are simple, the stability of the pair holds.

2. If $m \geq 2$ and eigenvalues are simple, by Hautus lemma we have

$$\text{Rank}\left[\mathbf{A} - \sigma_j \mathbf{I}_n \;\middle|\; \mathbf{B}\right] = n, \qquad \forall j \in \{1, ..., n\}.$$

   therefore, this condition is feasible if

$$\forall j \in \{1, \ldots, n\},\ \exists k \in \{1, \ldots, m\} :\ b_{jk} \neq 0.$$

On the stability of the pair $(\mathbf{A}, \mathbf{B})$

3. If $m \geq 1$ and eigenvalues are not simple, our Kalman's matrix $\mathbf{K}$ doesn't have complete rank. By picking

$$N_0 = \max\{d_1, \ldots, d_n\},$$

where d_j denote the algebraic multiplicity of the eigenvalue σ_j for $j \in \{1, \ldots, n\}$, we construct a matrix $\mathbf{N}$ such that

$$\tilde{\mathbf{K}} = [\mathbf{K} \,|\, \mathbf{N}],$$

has complete rank. Hence the controllability follows.


H. Lhachemi, I. Munteanu, and C. Prieur (2023)
Boundary output feedback stabilisation for 2-D and 3-D parabolic equations

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Manipulating saturations

We define the *deadzone nonlinearity* $\phi : \mathbb{R}^m \to \mathbb{R}^m$

$$\phi(u) = \mathfrak{sat}(u) - u, \quad \forall u \in \mathbb{R}^m,$$

therefore we have the equation $\mathfrak{sat}(u) = u + \phi(u)$, where we have the linear part and the geometric part.

Also, the following *generalized sector condition* holds

Lemma

If for some $\mathbf{C}, \mathbf{K} \in \mathbb{R}^{m\times n}$, $z \in \mathbb{R}^n$ *and* $j \in \{1, ..., m\}$ *it holds that* $|((\mathbf{K} - \mathbf{C})z)_j| \leq \ell$, *then*

$$\phi_j(\mathbf{K}z)(\phi_j(\mathbf{K}z) + (\mathbf{C}z)_j) \leq 0.$$

Then we can prove

Proposition

Let $\mathbf{K} \in \mathbb{R}^{m\times n}$ *such that* $\mathbf{A}+\mathbf{BK}$ *is Hurwitz. Let* $\mathbf{P} \in \mathbb{R}^{n\times n}$ *be symmetric positive definite,* $\mathbf{D} \in \mathbb{R}^{m\times m}$ *diagonal positive definite and* $\mathbf{C} \in \mathbb{R}^{m\times n}$ *such that*

$$\mathbf{M}_1 := \begin{pmatrix} (\mathbf{A}+\mathbf{BK})^\top \mathbf{P} + \mathbf{P}(\mathbf{A}+\mathbf{BK}) & \mathbf{PB} - (\mathbf{DC})^\top \\ (\mathbf{PB})^\top - \mathbf{DC} & -2\mathbf{D} \end{pmatrix} \prec 0, \tag{20}$$

and

$$\mathbf{M}_2 := \begin{pmatrix} \mathbf{P} & (\mathbf{K}-\mathbf{C})^\top \\ \mathbf{K}-\mathbf{C} & \ell^2 \mathbf{I}_m \end{pmatrix} \succeq 0. \tag{21}$$

Then the finite-dimensional system (15) *is locally exponentially stable in 0 with a region of attraction given by*

$$\mathcal{A} := \{z \in \mathbb{R}^n : \quad z^\top \mathbf{P} z \leq 1\}.$$

Proposition

Moreover, in $\mathcal{A}$ *the function* $V_1(z) := z^\top \mathbf{P} z$ *with* $z \in \mathbb{R}^n$ *decreases exponentially fast to* 0 *along the solutions of* (15)*, i.e, there is a constant* $\alpha > 0$ *so that*

$$\dot{V}_1(z) \leq -\alpha |z|^2, \quad z \in \mathcal{A}.$$

Sketch of the proof

Deriving over the trajectories of $V_1(z)$,

$$\begin{aligned}\dot{V}_1(z) &= z^\top \left[(\mathbf{A}+\mathbf{BK})^\top \mathbf{P} + \mathbf{P}(\mathbf{A}+\mathbf{BK})\right] z + 2z^\top \mathbf{PB}\phi(\mathbf{K}z) \\ &\leq z^\top \left[(\mathbf{A}+\mathbf{BK})^\top \mathbf{P} + \mathbf{P}(\mathbf{A}+\mathbf{BK})\right] z + 2z^\top \mathbf{PB}\phi(\mathbf{K}z) \\ &\quad - 2\phi(\mathbf{K}z)^\top \mathbf{D}(\phi(\mathbf{K}z) + \mathbf{C}z)) \\ &= \begin{pmatrix} z \\ \phi(\mathbf{K}z) \end{pmatrix}^\top \mathbf{M}_1 \begin{pmatrix} z \\ \phi(\mathbf{K}z) \end{pmatrix}.\end{aligned}$$

Since $\mathbf{M}_1$ is negative definite, we take $-\alpha < 0$ as the biggest eigenvalue of the matrix we get

$$\dot{V}_1(z) \leq -\alpha |z|^2.$$

Sketch of the proof

Now we need to include the region of attraction into the generalized sector condition:

$$\mathcal{A} = \{z \in \mathbb{R}^n \ : \ z^\top \mathbf{P} z \leq 1\} \subseteq \{z \in \mathbb{R}^n \ : \ |(\mathbf{K} - \mathbf{C})z| \leq \ell\},$$

this is equivalent to

$$\mathbf{M}_2 \setminus \ell^2 \mathbf{I}_m = \mathbf{P} - (\mathbf{K} - \mathbf{C})^\top \frac{1}{\ell^2} \mathbf{I}_m (\mathbf{K} - \mathbf{C}) \succeq 0.$$

where $\mathbf{M}_2 \setminus \ell^2 \mathbf{I}_m$ denote Schur's complement. By Schur's lemma we find that

$$\mathbf{M}_2 = \begin{pmatrix} \mathbf{P} & (\mathbf{K} - \mathbf{C})^\top \\ \mathbf{K} - \mathbf{C} & \ell^2 \mathbf{I}_m \end{pmatrix} \succeq 0.$$

which ends the proof.

Theorem (Stabilization in $L^2(0,L)$)

Consider system (11) *with* $(\mathbf{A},\mathbf{B})$ *stabilizable. Let* $\mathbf{K}$ *such that* $\mathbf{A}+\mathbf{BK}$ *is Hurwitz and* $K : X_n \to \mathbb{R}^m$ *be as* (16)*. Then the closed-loop system with feedback* K

$$\begin{cases} w_t(t,x) + w_{xxxx}(t,x) + \lambda w_{xx} = u, & t>0,\ x\in(0,L), \\ w(t,0) = w(t,L) = 0, & t>0 \\ w_x(t,0) = w_x(t,L) = 0, & t>0 \\ w(0,x) = w_0(x), & x\in(0,L). \end{cases}$$

where

$$u = \sum_{j=1}^{m} b_k(x)\mathfrak{sat}((K\pi_n w(t,\cdot))_k)$$

is locally exponentially stable in 0 with region of attraction $i(\mathcal{A})\times X_n^{\perp}$*. In addition, the constants of decay can be chosen uniformly on* $i(\mathcal{A})\times X_n^{\perp}$*.*

Stabilization in $H_0^2(0, L)$.

Using the same control as the last problem, it is possible to get a stabilization result for $H_0^2(0, L)$.



P. Guzmán, S. Marx and E. Cerpa (2019)
Stabilization of the linear Kuramoto-Sivashinsky equation with a delayed boundary control

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Stabilization in $H_0^2(0,L)$.

Objective: To prove that there exists $(C,\omega) \in (1,\infty)\times(0,\infty)$ such that

$$||w||^2_{H_0^2(0,L)} \leq Ce^{-\omega t}\,||w_0||^2_{H_0^2(0,L)}\,. \tag{22}$$

We will considerate the Lyapunov candidate function

$$V_2(t) = -\frac{1}{2}\langle Aw, w\rangle_{L^2(0,L)} + \frac{M}{2}V_1(t) = -\frac{1}{2}\sum_{j=1}^{\infty}\sigma_j w_j(t)^2 + \frac{M}{2}z^\top \mathbf{P} z,$$

where $\beta_{min}, \beta_{max} > 0$ are the min and max eigenvalues of $\mathbf{P}$ and $M > 0$.

Previous lemmas

Lemma

There are constants $C_1, C_2 > 0$ *such that*

$$\frac{C_1}{2}|z|^2 + \frac{C_1}{2C_2}||w||^2_{H_0^2(0,L)} \leq V_2(t), \qquad \forall t \geq 0. \tag{23}$$

Lemma

Exists $\omega > 0$ *such that*

$$V_2(t) \leq e^{-\omega t}\, V_2(0), \qquad \forall t \geq 0. \tag{24}$$

Lemma

Exists $C_4 > 0$ *such that*

$$V_2(0) \leq C_4||w_0||^2_{H_0^2(0,L)}. \tag{25}$$

Principal result

Theorem (Stabilization in $H_0^2(0,L)$.)

Let w be a solution of (11), z defined in (14) and the constants

$$C_1=\min\left\{\frac{M\beta_{min}-\sigma_1}{2},\frac{1}{2}\right\},\quad C_2=\max\left\{\lambda^2,2-\frac{\lambda^2}{\sigma_{n+1}}\right\},\quad C_4=\frac{1+M\beta_{max}C_p^2}{2},$$

and

$$C_3>\frac{1}{2\alpha},\quad M>\max\left\{\frac{\sigma_1}{\beta_{min}},-\frac{1}{\sigma_{n+1}},2C_3\beta_{max}\right\}.$$

Then it follows that

$$||w||^2_{H_0^2(0,L)}\leq\frac{2C_2C_4}{C_1}e^{-\omega t}\,||w_0||^2_{H_0^2(0,L)}\,, \tag{26}$$

with decay rate

$$\omega=\frac{1}{2C_3\beta_{max}}.$$

Thanks for your attention!

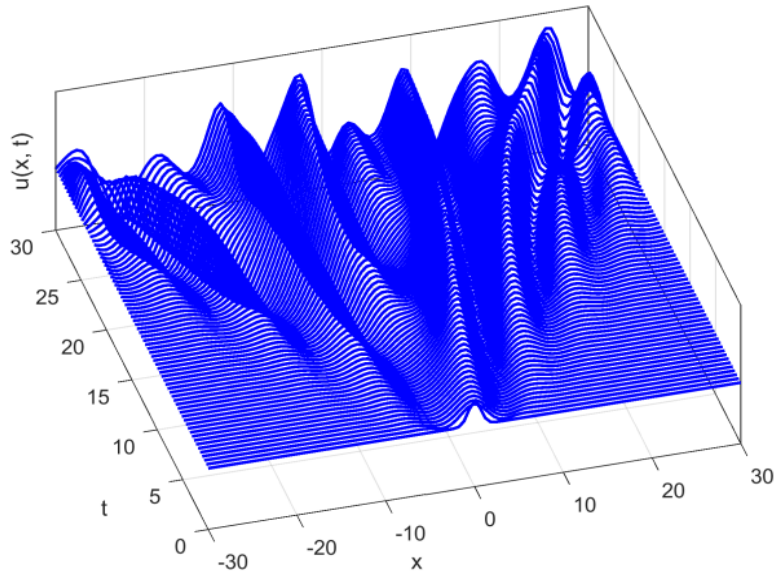


Figure: Numerical solution of non-linear KS in 3D using Runge-Kutta

H. Bhatt and A. Chowdhury (2019)
A compact fourth-order implicit-explicit Runge-Kutta type scheme for numerical solution of the Kuramoto-Sivashinsky equation.