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On the robustness of the stability for first-order hyperbolic systems with respect to different perturbations

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Chapter 1

Introduction

1.1 Motivation

The transfer function is one of the oldest and most successful tools in control theory. To fix ideas, consider the following canonical example: a mass $m > 0$ attached to a wall through a spring of stiffness $k > 0$ and a damper of coefficient $d > 0$, driven by an external force $U(t)$ applied to the mass. If $y(t)$ denotes the displacement of the mass from its equilibrium position, Newton's second law gives the following ordinary differential equation (ODE)

$$\begin{cases} m\,\ddot{y}(t) + d\,\dot{y}(t) + k\,y(t) = U(t), & t \geq 0, \\ y(0) = y_0, \\ \dot{y}(0) = 0, \end{cases} \tag{1.1}$$

Here $U(t)$ plays the role of the *input* of the system, an external action that we, as the observer or the controller, choose or measure independently of the system's internal dynamics, while $y(t)$ plays the role of the *output*, the quantity of interest, which we measure or wish to regulate, and whose evolution is entirely determined by (1.1) once $U(t)$ and the initial conditions are fixed. This input/output splitting is what distinguishes the transfer function point of view from simply solving the ODE: instead of describing the full trajectory $y(t)$ for one specific $U(t)$, one asks for the input-to-output map itself, as an object that can later be composed, inverted, or interconnected with other maps such as a feedback law, as for instance, $U(t) = -k_p\,y(t)$.

Taking the Laplace transform of (1.1) with zero initial conditions turns the differential equation into an algebraic relation $Y(s) = G(s)U(s)$, where $U(s), Y(s)$ denote the Laplace transforms of $U(t), y(t)$, respectively, and

$$G(s) = \frac{1}{ms^2 + ds + k} \tag{1.2}$$

is a rational function of the complex variable s, called the transfer function of the system. The idea of working directly with the input/output map $G(s)$, rather than with the full state trajectory, can be traced back to the work of Nyquist [Nyq32] and Bode [Bod45] who used the frequency response encoded in $G(i\omega)$ to assess closed-loop stability directly from the open-loop transfer function, without solving the underlying differential equation explicitly. This point of view, replacing a differential equation by an algebraic one in the Laplace domain and reading off stability from the location of the poles of $G(s)$, the roots of the characteristic polynomial $ms^2 + ds + k$ in (1.2), in the complex plane, became the backbone of what is now called classical control theory, and it is developed in detail in standard references such as Kailath [Kai80], Doyle, Francis and Tannenbaum [DFT92], Ogata [Oga10] and Åström and Murray [ÅM08].

When the underlying system is no longer described by an ODE but by a partial differential equation (PDE), the same formal procedure, taking the Laplace transform in time, solving the resulting boundary value problem in space, and dividing the transformed output by the transformed input, can still be carried out in some cases, but the outcome is qualitatively different: instead of a rational function of s, one typically obtains an irrational function built out of exponentials or hyperbolic functions of $\sqrt{s}$ or of s itself, with infinitely many poles and zeros in general. This extension of the transfer function method to distributed parameter systems has been systematically developed by Curtain and Zwart [CZ95] and surveyed, with a catalogue of worked PDE examples, by Curtain and Morris [CM09]. See Example 2.5 for an application on an ODE and Section 2.1 for a summary and for worked examples relevant to this thesis.

The present thesis is concerned with a particular class of such distributed parameter systems: linear 2×2 hyperbolic systems of balance laws on a bounded interval, controlled and observed at two boundary points that are anti-located. In this setting, the transfer function of the closed-loop system is a transcendental function of s, and exponential stability of the system is equivalent, via the spectral mapping theorem, to all the roots of the corresponding characteristic equation lying in an open left half-plane $\{\Re(s) < -\delta\}$ for some $\delta > 0$ (Theorem 2.14). Locating these roots, and tracking how they move under structural perturbations of the model, is the central technical problem addressed throughout this work.

1.2 Diffusive perturbations of hyperbolic systems and robustness

It is well known that boundary feedback laws that stabilize a hyperbolic system by compensating a delay as is necessarily the case when the control and the observation are anti-located can be extremely sensitive to arbitrarily small errors in the value of the transport velocity used to build the control law [Dat88]. Bastin, Coron and Hayat [BCH23; BCH25b] showed, for two structurally different 2×2 hyperbolic systems, the first being a pair of anti-located transport equations, and the second one a linearized Saint-Venant type density-flow system, that this apparent lack of robustness is in fact an artefact of neglecting a physically present, arbitrarily small second-order diffusion viscosity term: as soon as such a term is reinstated in the model, the same delayed output-feedback law that was designed for the inviscid system becomes robust, and the maximal spectral abscissa of the closed-loop system converges, as the viscosity $\eta \to 0^+$, to a computable value $-\ln(2)$ up to the relevant time and velocity scales that is uniform in η for η small. Chapters 3 and 4 of this thesis reproduce and organize these results in [BCH23], [BCH25b] respectively, giving in each case:

1. The computation of the single-channel transfer function $f_\eta(s)$ of the diffusively regularized transport equation, obtained by solving the Laplace-transformed second-order ODE in the space variable and imposing the appropriate boundary conditions,
2. The closed-loop characteristic equation obtained by combining $f_\eta(s)$ with the delayed feedback law, of the common form $f_\eta(s)^2 - f_\eta(s)e^{-s} - 1 = 0$,
3. An asymptotic analysis, as $\eta \to 0^+$, of the roots of this characteristic equation, carried out through purely complex-analytic arguments in Chapter 3 and refined, in Chapter 4, using topological degree theory (Section 2.3) to obtain a two-sided bound on the maximal spectral abscissa,

4. A robustness analysis showing that the conclusion persists under a further, independent perturbation of the nominal transport velocity.

Section 3.6 also exhibits a cautionary example, closely related to the main model, for which the same diffusive regularization has the opposite, destabilizing effect, showing that the stabilizing role of viscosity observed in [BCH23; BCH25b] is not a generic feature of diffusively regularized hyperbolic systems, but depends delicately on where the diffusion acts relative to the boundary coupling.

1.3 Fourth-order diffusion and the goal of this thesis

The results described above raise a natural structural question: are they special to second-order Laplacian-type diffusion, or do they persist under a higher-order dissipative regularization? This question is not merely of academic interest, since fourth-order dissipative terms of the form $\eta\,\partial_x^4$ arise naturally, for instance, as regularizations of thin-film and beam-type models, and their presence changes the qualitative nature of the underlying spectral problem: the characteristic equation for the single-channel transfer function goes from the biquadratic $\eta\lambda^2 - \lambda - s = 0$ of the second-order case, which is solved explicitly through the auxiliary variable $z = \sqrt{1+4\eta s}$, to the genuinely quartic equation $\eta\lambda^4 + \lambda + s = 0$, whose four roots no longer admit a simple closed-form expression amenable to the same change of variables. This loss of explicit solvability is the central obstruction that Chapter 5 is devoted to overcoming.

The goal of this thesis is therefore twofold:

1. To give a self-contained account, in Chapters 3 and 4, of how the transfer function method [BCH23; BCH25b] helps to prove that a small second-order diffusive term restores the robustness of delayed boundary-feedback stabilization for two representative 2×2 hyperbolic systems, together with the complementary example of Section 3.6 where the same mechanism destabilizes a closely related system.

2. To investigate, in Chapter 5, what happens when the second-order diffusion term is replaced by a fourth-order one, $\eta\,\partial_x^4$, in the coupled transport system of Chapter 3. Since an explicit formula for the transfer function is no longer available, we instead develop an asymptotic classification of the four characteristic roots as $\eta \to 0^+$, separating a finite branch, which recovers the inviscid transfer function e^{-s}, from three exploding branches responsible for a boundary-layer correction of order $\eta^{-1/3}$. We use this classification to derive an asymptotic expansion of the transfer function $f_\eta(s)$, first for s in compact sets and then along the frequency scaling $s = \sigma + i\eta^{-1/4}\psi$ relevant to the stability problem, and we use it, together with the topological degree argument of Chapter 4, to prove the analogue of the robustness result of Chapter 3 in this fourth-order setting.

1.4 Organization of the thesis

Chapter 2 collects, in a self-contained way and with worked examples, the four tools used throughout the thesis: transfer functions of ODE and PDE systems (Section 2.1), Lyapunov functionals for hyperbolic balance laws (Section 2.2), topological degree theory (Section 2.3), and a discussion, on an elementary transport equation, of when the Lyapunov and the transfer function approaches give the same stability threshold and why the latter can fail to be robust (Section 2.4). Chapter 3, presenting results from [BCH23], treats the diffusively regularized system of two anti-located transport equations, proves the associated

robustness result, and includes the destabilizing counterexample of Section 3.6. Chapter 4, presenting results from [BCH25b], treats the linearized density-flow of Saint-Venant type system under a second-order viscosity term, obtaining both an upper and a lower bound on the maximal spectral abscissa. The former following the ideas from Chapter 3 and the later using topological degree theory. Chapter 5, which contains the original contribution of this thesis, extends the analysis of Chapters 3 and 4 to a fourth-order diffusive regularization, and states the resulting robustness theorem together with the structural lemmas needed for its proof.

Chapter 2

Preliminaries

This chapter collects, in a self-contained manner, the three main tools used throughout the thesis: transfer functions of boundary-controlled distributed parameter systems (Section 2.1), Lyapunov functionals for hyperbolic balance laws (Section 2.2) and topological degree theory (Section 2.3). Finally, a discussion on an elementary transport equation, of when the Lyapunov and transfer function approaches give the same stability threshold and why the latter can fail to be robust (Section 2.4). Each subsection includes at least one worked example directly relevant to the systems studied in the chapters that follow.

2.1 Transfer functions of distributed parameter systems

In this section we define and motivate the idea behind using transfer functions in PDE problems. This is mainly an ODE-problem tool, where, by Laplace transform, we start from a differential equation and end up with an algebraic equation, which is always a rational function.

On the other hand, when we try to transform a PDE problem, we apply Laplace transform in time to get a differential equation in the space variable. After solving this ODE, we achieve an irrational function.

The main difference between rational and irrational functions appears when we study their poles and zeros. Irrational transfer functions often have infinitely many poles and zeros, as in the case of the functions

$$\frac{\cosh(\sqrt{s})}{\sinh\sqrt{2s}}, \qquad \frac{e^{-2s}}{s(1+e^{-s})}.$$

Many other functions as e^{-s} have neither poles nor zeros. In systems of PDEs, the location of the poles and zeros highly depends on the choice of boundary conditions, which will be discussed in Chapters 3 and 4.

During this section, we shall consider the following class of infinite-dimensional system with input u and output y:

$$\begin{cases} \dot{z}(t) = Az(t) + Bu(t), & t \geq 0, \\ y(t) = Cz(t) + Du(t), \\ z(0) = z_0. \end{cases} \tag{2.1}$$

We will denote by $\mathcal{B}(E, F)$ the space of bounded linear operators between two functional spaces E and F.

Definition 2.1 (State linear system [CZ95])**.** Let U, Y and Z be Hilbert spaces. $\Sigma(A, B, C, D)$ denotes the state linear system (2.1), where A is the infinitesimal generator of the strongly continuous semigroup $T(t) \in \mathcal{B}(Z, Z)$, $B \in \mathcal{B}(U, Z)$, $C \in \mathcal{B}(Z, Y)$ and $D \in \mathcal{B}(U, Y)$. We consider $\Sigma(A, B, C, D)$ for all initial states $z_0 \in Z$ and all inputs $u \in L^2([0, \tau]; U)$. The state z is the mild solution of (2.1),

$$z(t) = T(t)z_0 + \int_0^t T(t-s)Bu(s)ds, \quad 0 \leq t \leq \tau. \tag{2.2}$$

The relationship between the Laplace transform of the inputs and outputs is expressed in terms of a linear map known as the transfer function.

Definition 2.2 (Transfer function, [CZ95])**.** Let $\Sigma(A, B, C, D)$ be a linear, time-invariant control system with scalar input $U(t)$ and scalar output $Y(t)$, described by a well-posed initial-boundary value problem for a partial differential equation. Given a zero initial condition, let $U(s) = \mathcal{L}[U(t)](s)$, $Y(s) = \mathcal{L}[Y(t)](s)$ denote the Laplace transforms of the input and output signals. Whenever the relation $Y(s) = G(s)U(s)$ holds for s in a right half-plane $\Re(s) > \alpha$, the function G is called the transfer function of $\Sigma(A, B, C, D)$. In the same way, the impulse response h of $\Sigma(A, B, C, D)$ is the inverse Laplace transform of G. [CM09; CZ95].

Remark 2.3 (Assumption on initial data)**.** As in Theorem 2.2, every time a transfer function is computed in this thesis by taking the Laplace transform of a partial differential equation in time, we assume that the system starts from rest, i.e., that the initial condition is identically zero $y(0, \cdot) \equiv 0$, unless explicitly stated otherwise. For this reason, the initial condition is systematically omitted from the statement of the initial-boundary value problems considered in Chapters 3, 4 and 5, which describe only the input-output map of the corresponding system. When a system is instead analyzed for a general, non-zero initial condition (as in Section 2.4 below, where both a Lyapunov and a transfer-function proof are compared), this is indicated explicitly.

Remark 2.4. In contrast with the finite-dimensional (ODE) case, transfer functions of PDE systems are typically *irrational* functions of s (exponentials, hyperbolic functions, or ratios thereof), and the classical rational-function notions of pole, zero, and relative degree must be adapted. A careful discussion of these subtleties, together with a catalogue of worked examples is given in [CM09]. The mathematical justification of formally taking Laplace transforms of the boundary value problem is discussed in [CZ95, Ch. 4] and in [Mor20].

First we want to address the usefulness of transfer functions in models of ordinary differential equations.

Example 2.5 (Transfer function of a loudspeaker [CM09])**.** The loudspeaker diaphragm is modeled as a mass–spring–damper system driven by an electrical voltage and loaded by the acoustic pressure of the duct it drives:

$$m\ddot{x}(t) + d\dot{x}(t) + kx(t) = B\,V(t) - A\,p(t, 0), \tag{2.3}$$

together with the volume-flow coupling condition between the diaphragm and the duct,

$$A\,\dot{x}(t) = \pi a^2\, v(t, 0). \tag{2.4}$$

Here $x(t)$, $V(t)$, m, d, k, B, A, $p(t,0)$, $v(t,0)$ and a correspond, respectively, to:

$x(t)$: displacement of the loudspeaker diaphragm (mechanical output).

$V(t)$: driving voltage applied to the loudspeaker coil (control input).

m: effective mass of the diaphragm.

d: mechanical damping coefficient.

k: stiffness of the diaphragm suspension.

B: electro-mechanical force factor, $B = B\ell/R_{\text{coil}}$, where $B\ell$ is the motor constant (magnetic flux density times coil wire length) and R_{coil} is the coil resistance. At low frequencies the coil inductance is neglected, so the generated force is approximately proportional to $V(t)$ through this constant.

A: effective area of the diaphragm.

$p(t,0)$: acoustic pressure at the duct end $x = 0$, which acts as a reaction load on the diaphragm.

$v(t,0)$: particle velocity of the air at $x = 0$, which couples the diaphragm motion to the duct.

a: radius of the duct, so that πa^2 is its cross-sectional area.

This model arises as a low-frequency approximation of a wave equation. Now we want to calculate the transfer function of the system.

Taking Laplace transforms of (2.3) with zero initial conditions ($x(0) = \dot{x}(0) = 0$), and using $\mathcal{L}\{\ddot{x}(t)\}(s) = s^2 x(s)$, $\mathcal{L}\{\dot{x}(t)\}(s) = sx(s)$,

$$\Big(ms^2 + ds + k\Big)\, x(s) = B\, V(s) - A\, p(s,0). \tag{2.5}$$

Similarly, taking Laplace transforms of (2.4),

$$A\, s\, x(s) = \pi a^2\, v(s,0) \quad \Longrightarrow \quad x(s) = \frac{\pi a^2}{A\, s}\, v(s,0). \tag{2.6}$$

Substituting (2.6) into (2.5), we eliminate $x(s)$,

$$\Big(ms^2 + ds + k\Big)\, \frac{\pi a^2}{A\, s}\, v(s,0) = B\, V(s) - A\, p(s,0).$$

Solving for $Ap(s,0)$,

$$A\, p(s,0) = B\, V(s) - Z_0(s) v(s,0), \tag{2.7}$$

where

$$Z_0(s) := \frac{\pi a^2}{A\, s}\, \Big(ms^2 + ds + k\Big).$$

Setting $V(t) \equiv 0$ in (2.7) isolates the ratio of output pressure to input velocity, which defines the mechanical impedance of the loudspeaker, so we can identify the transfer function

$$Z_0(s) = \frac{p(s,0)}{v(s,0)}\bigg|_{V\equiv 0} = \frac{\pi a^2}{A\,s}\left(ms^2 + ds + k\right). \tag{2.8}$$

Thus $Z_0(s)$ is the transfer function from the input $v(t,0)$ (duct-side velocity) to the output $p(t,0)$ (duct-side pressure), obtained by dividing the Laplace transform of the output by that of the input. We shall notice at this point that the transfer function has only one pole given by $s = 0$ which says that the system is unstable since its real part is not negative.

Now we give some examples about the same method applied to partial differential equations.

Example 2.6 (Transport equation with delay, [BCH23])**.** Consider the elementary transport equation on $[0,1]$ with speed $v > 0$,

$$\begin{cases} \partial_t y(t,x) + v\,\partial_x y(t,x) = 0, & (t,x) \in \mathbb{R}_+ \times (0,1), \\ y(t,0) = U(t), & t \in \mathbb{R}_+, \\ Y(t) = y(t,1), & t \in \mathbb{R}_+. \end{cases}$$

Taking the Laplace transform in time yields the ODE

$$\begin{cases} s\,\hat{y}(s,x) + v\,\partial_x \hat{y}(s,x) = 0, \\ \hat{y}(s,0) = U(s), \\ Y(s) = \hat{y}(s,1), \end{cases}$$

whose solution is $\hat{y}(s,x) = \hat{y}(s,0)e^{-sx/v} = U(s)e^{-sx/v}$. Hence the transfer function from $U(s)$ to $Y(s)$ is

$$G(s) = e^{-s\tau}, \qquad \tau = \frac{1}{v},$$

a pure delay of length τ. In particular, this transfer function encodes $Y(t) = U(t-\tau)$, so information injected at $x = 0$ reappears unchanged at $x = 1$, after the transport time τ.

Example 2.7 (Heat equation, [CM09])**.** Consider the heat equation concerning a rod of length L

$$C_p\rho\partial_t z(t,x) = K_0\partial_{xx}^2 z(t,x), \quad t \geq 0,\ x \in (0,L). \tag{2.9}$$

Here, K_0 denotes the thermal conductivity, ρ is the mass density and C_p is the specific heat.

Suppose first that we are in the Neumann boundary control case, i.e., equation (2.9) with boundary conditions

$$\partial_x z(t,0) = 0, \qquad K_0\partial_x z(t,L) = U(t), \tag{2.10}$$

where $U(t)$ denotes the rate of heat flow into the rod. The temperature is measured at a fixed point $x_0 \in (0,L]$ on the rod, so the observation is given by

$$Y(t) = z(t,x_0). \tag{2.11}$$

Taking the Laplace transform on (2.9), we achieve the system

$$\begin{cases} C_p\rho s\partial_t z(s,x) = K_0\partial^2_{xx} z(s,x), \\ \partial_x z(s,0) = 0, \\ K_0\partial_x z(s,L) = U(s), \\ Y(s) = z(s,x_0). \end{cases} \tag{2.12}$$

The general solution for (2.12) is

$$z(s,x) = A\sinh\left(\frac{\sqrt{s}x}{\alpha}\right) + B\cosh\left(\frac{\sqrt{s}x}{\alpha}\right), \quad \alpha^2 := \frac{K_0}{C_p\rho}.$$

After applying the boundary conditions (2.10), we obtain

$$z(s,x) = \frac{\alpha\cosh\left(\frac{\sqrt{s}x}{\alpha}\right)}{K_0\sqrt{s}\sinh\left(\frac{\sqrt{s}L}{\alpha}\right)}U(s),$$

so finally the transfer function $G_1(s,x_0)$ takes the form

$$G_1(s,x_0) = \frac{Y(s)}{U(s)} = \frac{\alpha\cosh\left(\frac{\sqrt{s}x_0}{\alpha}\right)}{K_0\sqrt{s}\sinh\left(\frac{\sqrt{s}L}{\alpha}\right)}.$$

Some remarks are in order. First note that $G_1(s,x_0)$ depends on the point from which we are observing the problem. On the other hand, the poles of G_1 are the zeros of the denominator, given by $-(k\pi\alpha/L)^2$, while the zeros of G_1 are the zeros of the numerator, given by $-((k\pi+\pi/2)\alpha/x_0)^2$. Since we have a pole at 0, $G_1(s,x_0)$ is not stable.

Suppose now that we have Dirichlet boundary control, i.e., equation (2.12) with boundary conditions

$$z(t,0) = 0, \qquad z(t,L) = U(t). \tag{2.13}$$

We still measure the temperature at a point x_0 as in (2.11). After taking Laplace transform and solving the system, we get the transfer function

$$G_2(s,x_0) = \frac{\sinh\left(\frac{\sqrt{s}x_0}{\alpha}\right)}{\sinh\left(\frac{\sqrt{s}L}{\alpha}\right)}.$$

Now note that the zero at $s=0$ of the numerator cancels with the denominator. Thus the poles of $G_2(s,x_0)$ are the same as in $G_1(s,x_0)$, except that there is no pole at $s=0$.

$G_2(s,x_0)$ is analytic and uniformly bounded in the open right half-plane, and since the poles are all in this set, $G_2(s,x_0)$ is stable.

Notice that in both situations we obtain an irrational transfer function built out of hyperbolic cosines.

2.2 Lyapunov functions for hyperbolic systems

Lyapunov functions are a strong tool to study stabilization problems involving ODEs and PDEs. Here we will focus on the PDE case.

The main idea is to consider an energy functional-type $V(t)$ for a conserved quantity of the PDE. After taking the derivative and performing some calculations, we expect to find an inequality involving $\dot{V}(t)$ and $-\kappa V(t)$, where $\kappa > 0$. This allows us to use Grönwall inequality and achieve the desired inequality $V(t) \leq e^{-\kappa t}V(0)$.

Example 2.8 (Heat equation)**.** Suppose that we have the system

$$\begin{cases} y_t(t,x) - y_{xx}(t,x) = 0, & (t,x) \in \mathbb{R}_+ \times (0,1), \\ y(t,0) = y(t,1) = 0, & t \in \mathbb{R}_+, \\ y(0,x) = y_0(x), & x \in (0,1), \end{cases} \tag{2.14}$$

where $y_0 \in L^2(0,1)$. Consider the Lyapunov functional,

$$V(t) = \frac{1}{2}\int_0^1 |y(t,x)|^2 dx = \frac{1}{2}\|y(t)\|^2_{L^2(0,1)}. \tag{2.15}$$

This is the usual Dirichlet energy. Suppose that we have enough regularity. Then, taking the time derivative and using integration by parts, we have

$$\begin{aligned} \dot{V}(t) &= \int_0^1 y(t,x)y_t(t,x)dx \\ &= \int_0^1 y(t,x)y_{xx}(t,x)dx \\ &= [yy_x]\,|_0^1 - \int_0^1 |y_x(t,x)|^2 dx \\ &= -\int_0^1 |y_x(t,x)|^2 dx \\ &\leq -C_p V(t), \end{aligned}$$

where we used the Poincaré inequality and $C_p > 0$ denotes the Poincaré constant. By Grönwall inequality, we get

$$V(t) \leq e^{-C_p t}V(0), \qquad \forall t \geq 0,$$

which in terms of L^2-norm becomes

$$\|y(t)\|_{L^2(0,1)} \leq e^{\frac{-C_p t}{2}}\|y_0\|_{L^2(0,1)}, \qquad \forall t \geq 0.$$

This proves the exponential stability in L^2-norm for the system.

In this thesis we want to address results on hyperbolic systems of PDEs, where the natural prototype of Lyapunov functional will be explained. First we need the concept of Riemannian coordinates.

Definition 2.9 (Riemann coordinates)**.** Let $\partial_t w(t,x) + A(x)\partial_x w(t,x) = 0$, $x \in (0,L)$, be a 2×2 system that is strictly hyperbolic, i.e., $A(x)$ has two real, distinct eigenvalues $-\mu_1(x) < 0 < \mu_2(x)$ for every x, with associated left eigenvectors $\ell_1(x), \ell_2(x)$. The Riemann coordinates of the system are $R_i(t,x) := \ell_i(x) \cdot w(t,x)$, $i = 1,2$. By construction, in these coordinates the system decouples, to leading order, into two pure transport equations along the characteristics,

$$\partial_t R_1 + \lambda_1(x)\partial_x R_1 = 0, \qquad \partial_t R_2 - \lambda_2(x)\partial_x R_2 = 0,$$

with $\lambda_i(x) = \mu_i(x) > 0$, each variable being transported with its own characteristic speed, any coupling between the two components of the original system is then relegated entirely to the boundary conditions.

Definition 2.10 (Basic quadratic Lyapunov function, [BC16])**.** Consider a 2×2 hyperbolic system with Riemann coordinates (R_1, R_2) on $[0, L]$, with transport speeds $\lambda_1, \lambda_2 > 0$,

$$\begin{cases} \partial_t R_1 + \lambda \partial_x R_1 = 0, \\ \partial_t R_2 - \lambda_2 \partial_x R_2 = 0, \\ R_1(t,0) = k_1 R_2(t,0), \\ R_2(t,L) = k_2 R_1(t,L), \\ R_1(0,x) = R_{1,0}(x), \\ R_2(0,x) = R_{2,0}(x). \end{cases}$$

A candidate Lyapunov functional of *basic quadratic* type is

$$V(t) = \int_0^L \left[\frac{p_1}{\lambda_1} R_1^2(t,x)\, e^{-\mu x/\lambda_1} + \frac{p_2}{\lambda_2} R_2^2(t,x)\, e^{+\mu x/\lambda_2} \right] dx, \tag{2.16}$$

where $p_1, p_2, \mu > 0$ are free tuning parameters [BC16, Section 2.1]. The system is said to admit V as a strict Lyapunov function if $\dot{V}(t) \le -\delta V(t)$ along classical solutions, for some $\delta > 0$ independent of the initial data.

Remark 2.11. The main idea is to properly select the tuning parameters $p_1, p_2, \mu > 0$ in order to achieve the exponential stability. This highly depends on the boundary conditions used for the problem and it becomes a hard problem when non-linearities are included in the problem.

Theorem 2.12 (Sufficient stability condition, [BC16, Thm. 2.9])**.** *Consider a 2×2 hyperbolic system in Riemann coordinates (Theorem 2.9) with boundary conditions $R_1(t,0) = k_1 R_2(t,0) + k_3 X(t)$, $R_2(t,L) = k_2 R_1(t,L)$, coupled to a finite-dimensional actuator state $X(t) \in \mathbb{R}$, i.e., an auxiliary variable, generated by an internal dynamic controller and evolving according to its own ODE $\dot{X}(t) = f(X(t), R_1(t,0), R_2(t,L))$, that is fed into the boundary condition of the PDE and whose own dynamics must be stabilized jointly with those of the PDE; see [BC16, Ch. 2] for the precise form of f used here. If*

$$|k_2| < 1, \qquad |k_1 k_2| < 1, \qquad k_3 > 0,$$

then there exist $p_1, p_2, \mu, b > 0$ such that the functional

$$V(t) = \int_0^L \left[\tfrac{p_1}{\lambda_1} R_1^2 e^{-\mu x/\lambda_1} + \tfrac{p_2}{\lambda_2} R_2^2 e^{\mu x/\lambda_2} \right] dx + b X^2(t),$$

is a strict Lyapunov function, and the solution converges exponentially to zero in the L^2-norm.

Remark 2.13. Here we consider another tuning parameter $b > 0$. This is sometimes used as a function $b = b(t)$ to properly control boundary terms from which we do not have much information to control them.

Remark 2.14. By the spectral mapping theorem (see Theorem 3.1) for this class of boundary-coupled hyperbolic systems, exponential decay of a Lyapunov functional equivalent to the squared L^2-norm is equivalent to the poles of the associated transfer function lying in an open left half-plane $\{\Re(s) < -\delta\}$ for some $\delta > 0$ [BC16; CZ95]. This equivalence is what allows the two viewpoints of Section 2.1 (spectral abscissa of the characteristic equation) and Section 2.2 (decay of a quadratic functional) to be used interchangeably in the chapters that follow. When the spectral mapping property itself is not known a priori, the Gearhart–Prüss theorem provides an alternative route from a resolvent bound to semigroup decay [CZ95].

2.3 Topological degree theory

In the context of stability of hyperbolic systems, this is the newest tool among the ones applied in this thesis. The relationship between control theory and topological degree theory comes from the study of global controllability. If we know that a quadratic control system is locally controllable, then any system which is a perturbation of this quadratic control system by a linear term is globally controllable [Cor07, Sec. 3.5]. In our case, the main idea is to prove that there are many solutions for a transcendental characteristic equation arising from a global transfer function which models the phenomenon. Here we collect some properties about the theory following the exposition of [Cor07, Appendix B].

Theorem 2.15 (Existence and properties of the Brouwer degree, [Cor07, Thm. B.1])**.** *Let $\mathcal{D}$ denote the set of triples (φ, Ω, b) with $\Omega \subset \mathbb{R}^n$ nonempty, bounded and open, $\varphi \in C^0(\overline{\Omega}; \mathbb{R}^n)$ and $b \notin \varphi(\partial\Omega)$. There exists a unique map $\deg : \mathcal{D} \to \mathbb{Z}$, $(\varphi, \Omega, b) \mapsto \deg(\varphi, \Omega, b)$, satisfying:*

(i) $\deg(\mathrm{Id}_n|_\Omega, \Omega, b) = 1$ *for every* $b \in \Omega$.

(ii) (additivity) for disjoint open $\Omega_1, \Omega_2 \subset \Omega$ *with* $b \notin \varphi(\Omega \setminus (\Omega_1 \cup \Omega_2))$, $\deg(\varphi, \Omega, b) = \deg(\varphi|_{\Omega_1}, \Omega_1, b) + \deg(\varphi|_{\Omega_2}, \Omega_2, b)$.

(iii) (continuity) $\deg(\varphi, \Omega, b)$ *is locally constant with respect to uniform perturbations of* φ.

(iv) $\deg(\varphi, \Omega, b) = \deg(\varphi - b, \Omega, 0)$.

Moreover, if $\varphi \in C^1(\Omega)$ and every $x \in \varphi^{-1}(b)$ is a regular point, i.e, $\det(\varphi'(x)) \neq 0$, then $\deg(\varphi, \Omega, b) = \sum_{x \in \varphi^{-1}(b)} \operatorname{sign} \det \varphi'(x)$.

Proposition 2.16 (Homotopy invariance, [Cor07, Prop. B.8])**.** *Let $\Omega \subset \mathbb{R}^n$ be nonempty and open, $H \in C^0([0,1] \times \Omega; \mathbb{R}^n)$, and $b \in \mathbb{R}^n$ with $H(t,x) \neq b$ for all $(t,x) \in [0,1] \times \partial\Omega$. Then $\deg(H(0,\cdot), \Omega, b) = \deg(H(1,\cdot), \Omega, b)$.*

Proposition 2.17 (Existence of a preimage, [Cor07, Prop. B.10])**.** *Let $(\varphi, \Omega, b) \in \mathcal{D}$ be such that $b \notin \varphi(\Omega)$. Then $\deg(\varphi, \Omega, b) = 0$. Equivalently (contrapositive form): if $\deg(\varphi, \Omega, b) \neq 0$, there exists $x \in \Omega$ with $\varphi(x) = b$.*

Remark 2.18. Proposition 2.17 is the single most important consequence of degree theory used in this document: it converts a *computable, robust integer invariant* (the degree, which by Theorem 2.16 is insensitive to continuous deformations of the map, as long as no zero

crosses the boundary of the domain during the deformation) into an *existence statement* for a genuine solution of a nonlinear equation, without requiring any monotonicity, convexity or smoothness beyond continuity. It is used in Chapter 4 to locate a pole of a perturbed characteristic equation.

Remark 2.19. In the following example we will use the topological degree for continuous functions instead. This is well defined since every continuous map is homotopic to a differentiable one by Whitney's approximation Theorem, [Lee13, Theorem 6.26], where we can apply Theorem 2.15 and by homotopic invariance, Proposition 2.16, the degree of both maps coincide.

Example 2.20 (Locating a pole via the degree)**.** We will solve $x = \cos(x)$ in $\mathbb{R}$ using the topological degree theory. Define $g(x) = x - \cos(x)$, which is $C^\infty(\mathbb{R})$. We seek for $\overline{x}$ with $g(\overline{x}) = 0$.

First we fix the interval $(0, \pi)$. Note that

$$g(0) = -1 < 0, \qquad g(\pi) = \pi + 1 > 0.$$

Both are non-zero, so by Remark 2.19 the degree is well defined and

$$\deg(g, (0, \pi), 0) = \frac{1}{2}\left[\operatorname{sgn}(g(\pi)) - \operatorname{sgn}(g(0))\right] = 1 \neq 0.$$

By the existence property, Proposition 2.17, there exists $\overline{x} \in (0, \pi)$ such that $g(\overline{x}) = 0$, which means that $\overline{x} = \cos(\overline{x})$.

Furthermore, the degree itself, together with the regular point formula of Theorem 2.15, already gives uniqueness of $\overline{x}$ in $(0, \pi)$. Note that

$$g'(x) = 1 + \sin(x) > 0, \qquad \forall x \in (0, \pi),$$

since the only zeros of g' are $x = -\frac{\pi}{2} + 2\pi k$, $k \in \mathbb{Z}$, none of which lie in $(0, \pi)$. Hence every $x \in g^{-1}(0) \cap (0, \pi)$ is automatically a regular point, and the regular point formula of Theorem 2.15 applies to the whole preimage at once,

$$\deg(g, (0, \pi), 0) = \sum_{x \in g^{-1}(0) \cap (0,\pi)} \operatorname{sgn}(g'(x)) = \#(g^{-1}(0) \cap (0, \pi)),$$

where the last equality uses that $g'(x) > 0$ for $x \in (0, \pi)$, so every term in the sum equals $+1$. Since we already computed $\deg(g, (0, \pi), 0) = 1$, this forces $\#(g^{-1}(0) \cap (0, \pi)) = 1$: the zero $\overline{x}$ is the only zero of g in $(0, \pi)$, and it is automatically non-degenerate, i.e., $g'(\overline{x}) > 0$.

Remark 2.21. For another application where a non-local term appears in a conservation law, we refer to [CW13], where a homotopy argument is applied to connect the original characteristic equation where we do not have information about the poles, and an easier one where we already know the poles. The argument ends after applying the invariance under homotopy for the topological degree Proposition 2.16.

2.4 Perspective on the methods

Here we follow [ZP17], where the authors claim that the system

$$\begin{cases} \partial_t y(t,x) + \partial_x y(t,x) = 0, & x \in (0,1),\ t \geq 0, \\ y(t,0) = k y(t,1), & t \geq 0, \\ y(0,x) = y_0(x), & x \in (0,1), \end{cases} \tag{2.17}$$

is exponentially stable with the L^2 norm if and only if $|k| < 1$, see [ZP17, Theorem 1]. This differs from the original problem since we are using only one equation. The objective is to prove the result by Lyapunov functionals as in Section 2.2 and by transfer functions as in Section 2.1. First suppose that $|k| < 1$ and let $\mu > 0$ be a tuning parameter to be determined. Consider the Lyapunov functional candidate

$$V(t) = \int_0^1 |y(t,x)|^2 e^{-\mu x} dx.$$

Assuming enough regularity, we can derive the functional through the trajectories of (2.17) and use integration by parts to get

$$\begin{aligned} \dot{V}(t) &= 2\int_0^1 y(t,x) y_t(t,x) e^{-\mu x} dx \\ &= -2\int_0^1 y(t,x) y_x(t,x) e^{-\mu x} dx \\ &= \left[-|y(t,x)|^2 e^{-\mu x}\right] \Big|_{x=0}^{x=1} + \int_0^1 -\mu |y(t,x)|^2 e^{-\mu x} dx \\ &= (|k|^2 - e^{-\mu})|y(t,1)|^2 - \mu V(t). \end{aligned}$$

Selecting $\mu > 0$ such that $|k|^2 - e^{-\mu} < 0$, we get $\dot{V}(t) \leq -\mu V(t)$. Grönwall inequality yields the desired exponential stability for the Lyapunov functional,

$$V(t) \leq e^{-\mu t} V(0).$$

Using that $e^{-\mu} \leq e^{-\mu x} \leq e^0$ for any $x \in [0,1]$, we finally get the L^2 exponential stability,

$$\|y(t,\cdot)\|^2_{L^2(0,1)} \leq e^{\mu(1-t)} \|y_0\|_{L^2(0,1)}.$$

To prove the other implication we will use the contrapositive statement. Suppose that $|k| \geq 1$. By the method of characteristics, we know that the solutions of (2.17) are constant along the characteristics, which means that $x - t = c$ for $c \in \mathbb{R}$. We will use then that $y(t,x) = y_0(x-t) = c$. Let $t \in [n, n+1)$ where $n \in \mathbb{N}$. Then by the boundary condition we know that the dynamics of the solution yield

$$y(n,x) = k^n y_0(x), \quad \forall x \in (0,1),\ n \in \mathbb{N}.$$

Now we separate two cases, which will end the proof.

1. If $|k| = 1$, then $\|y(n,\cdot)\|_{L^2(0,1)} = \|y_0\|_{L^2(0,1)}$ for any $n \in \mathbb{N}$. This means that the solution is not exponentially stable for the L^2-norm.

2. If $|k| > 1$, we get that $\|y(n,\cdot)\|_{L^2(0,1)} = |k|^n \|y_0\|_{L^2(0,1)}$ cannot be exponentially stable for the L^2-norm.

Now we will tackle the problem using transfer functions. As per Theorem 2.3, this computation is carried out for zero initial data, i.e., $y(0,\cdot) \equiv 0$, even though (2.17) was stated, and solved above by the Lyapunov method, for a general $y_0 \in L^2(0,1)$. Applying Laplace transform in time to (2.17) with $y(0,\cdot) \equiv 0$, we get the equation $sy(s,x) + y_x(s,x) = 0$ with homogeneous solution being

$$y(s,x) = A(s)e^{-sx}. \tag{2.18}$$

Since $y(s,0) = ky(s,1)$, we get from (2.18) that

$$A(s)(1 - ke^{-s}) = 0.$$

Since we expect a non-zero solution, we ask $1 - ke^{-s} = 0$ to be true, which yields $k = e^s$, where $s \in \mathbb{C}$. To prove the stability of the system, we need the supremum of the real part of s to be negative. Taking the natural logarithm with the usual branch, yields the equation

$$s_n = \ln(|k|) + (\arg(k) + 2\pi n)i, \quad n \in \mathbb{Z},$$

where $\Re(s_n) = \ln(|k|)$. This is constant with respect to $n \in \mathbb{Z}$, so $\sigma_n = \sup_n \Re(s_n) = \ln(|k|) < 0$ if and only if $|k| < 1$. This is the expected exponential stability for the L^2-norm.

Remark 2.22. The transfer function computation above only establishes, strictly speaking, the exponential decay of the solution issued from $y_0 \equiv 0$. What makes it equivalent to the Lyapunov conclusion, valid for every $y_0 \in L^2(0,1)$, is that (2.17) generates a C_0-semigroup $(T(t))_{t\geq 0}$ on $L^2(0,1)$, and that the abscissa of convergence of the transfer function computed from zero initial data coincides with the spectral bound $s(A)$ of its infinitesimal generator A, which by the spectral mapping theorem (Theorem 3.1 below, applicable here since transport semigroups are eventually norm-continuous) equals the growth bound $\omega_0(T)$ governing the decay of $\|T(t)y_0\|$ for every $y_0 \in L^2(0,1)$, not just for zero initial data.

The previous discussion shows that in some cases, using transfer equations or Lyapunov analysis gives the same result. The problem, as we will see in Chapter 3, is the lack of robustness of the argument. This differs from the Lyapunov functionals approach, since in that case we only need to adjust the proper bounds to get a robustness result for the given perturbed problem.

Chapter 3

Stability and robustness for a hyperbolic system with boundary coupling

This Chapter is based on [BCH23].

3.1 Introduction

This chapter concerns a result of *output-feedback stabilization* of single-input-single-output (SISO) type for a system of hyperbolic equations. This problem has mostly been studied when the control and the observation are located on the same side of the boundary.

The contrast that this chapter contributes is that the control and the observation are located on opposite sides of the boundary, which complicates the problem. In particular, the input-output operator ceases to be a passive operator. It is also important to mention that the use of feedback laws for delay problems can be problematic, since they can be strongly sensitive to the uncertainties in the characteristic velocities of the underlying problem. This occurs when the design of the control depends on exact knowledge of a characteristic velocity that must be compensated by the control law, so that stability can be destroyed by arbitrarily small perturbations. This eigenvalue-sensitivity phenomenon is well documented for general time-delay systems [HV93; MN07; Fri14].

The literature on boundary control of hyperbolic systems offers several variations on where control and observation are placed and on which tool is used to establish robustness. When control and observation sit on the same side of the boundary, Aamo [Aam13] rejects disturbances in a 2×2 linear hyperbolic system through a backstepping design, and Anfinsen and Aamo [AA17b] treat disturbance rejection for a general heterodirectional collocated system subject to a boundary perturbation. The anti-located configuration studied in this chapter is comparatively less standard. The separation-principle design of Krstic, Guo, Balogh and Smyshlyaev [Krs+08] for the anti-located stabilization of an unstable wave equation is one of the earliest instances of this setting where the main tools are a backstepping design and a appropriate Lyapunov functional, and Anfinsen and Aamo [AA17a] stabilize a 2×2 system with anti-collocated sensing and control and, moreover, with unknown transport velocities $\lambda(x)$, $\mu(x)$, combining a backstepping transformation with a Lyapunov argument to obtain an adaptive law. A related source of loss of robustness is an unknown perturbation entering the dynamics through an integral, which is a non-local term. Bernard and Krstic [BK14] address exactly this situation, again via backstepping and Lyapunov techniques. The dissipative boundary condition Lyapunov functionals underlying much of this literature, and used throughout Chapter 2, trace back to Coron, Bastin and d'Andréa-Novel [CBd08]. Finally, on the application side, Dos Santos Martins and Prieur [DP08] study the non-linear Saint-Venant system for open-channel flow, complementing their theoretical boundary control design with numerical and experimental validations. The linearized density-flow system considered in Chapter 4 is closely related to this model.

Instead of a state-space Lyapunov argument, we work directly with the transfer function of the closed-loop system in the Laplace domain, since stability is equivalent, via the spectral mapping theorem for linear hyperbolic systems [Lic08], to the location of the poles of this transfer function, which lets us track explicitly how a small dissipative term shifts the location of the poles.

The goal of this chapter is to highlight some features of this problem, by developing the stabilization of an open-loop unstable system built from two coupled transport equations and a control-observation pair that are anti-located, i.e., they sit on different sectors of the boundary.

The chapter is organized as follows. Section 3.2 formulates the control problem, where it is first shown that the open-loop control system is unstable and cannot be stabilized by a simple feedback law proportional to the output of the system. It is then shown that the system can be stabilized by a dynamic control that involves an output-feedback law with delay. The problem with this control is that it is not robust with respect to perturbations of the delay, since the control requires exact knowledge of the transport velocity at all times, which is not achievable in practice.

The main contribution of the chapter is to prove that the robustness of the control against delay perturbations is recovered as soon as a small diffusion parameter is present in the system [BCH23]. To this end, Section 3.3 studies the system, first assuming that the plant under study is subject to a slight diffusion phenomenon interpreted as a viscosity. In this way, the (unstable) transfer function can be computed. Section 3.4 proves that the dynamic output law, which is not robust, and which is built for the case without viscosity, also stabilizes the viscous system when the unknown diffusion is small, and in that case the control turns out to be perfectly robust even when the diffusion is almost negligible. An interesting fact in this respect is that there is an upper bound for the decay rate once a small viscosity is included, a bound that is uniform with respect to the diffusion parameter η when it is small, while for the unperturbed system with $\eta = 0$ the decay rate is infinite. In this last case we say that the system is finite-time stable.

The analysis of Sections 3.4 and 3.5 shows that the effect of diffusion on stability is far from an obvious problem, contrary to what one might expect. In hyperbolic systems, the presence of diffusion or friction can either stabilize or destabilize the system. This last issue is discussed further in Chapter 4, where we present the example of another simple hyperbolic system that, in some respects, simplifies the previous case, and yet the diffusion term destroys stability instead of strengthening it.

3.2 Description of the Control Problem

The system that we study is made up of two transport equations, whose dynamics are described by the problem

$$\begin{cases} \partial_t y_1(t,x) + v\partial_x y_1(t,x) = 0, \\ \partial_t y_2(t,x) + v\partial_x y_2(t,x) = 0, \\ y_1(t,0) = y_2(t,1) + U(t), \\ y_2(t,0) = y_1(t,1), \\ Y(t) = y_1(t,1). \end{cases} \tag{3.1}$$

Here $U(t)$ is the control input and $Y(t)$ is the measured output of the system. In the classical sense, $v > 0$ denotes the transport velocity of each equation. This model can be illustrated as a plant, as we can see in Figure 3.1,

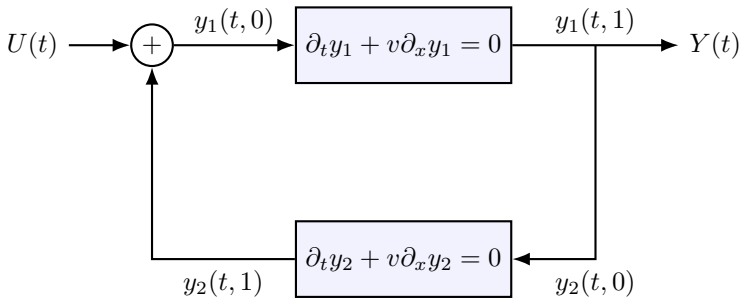


Figure 3.1: Plant (3.1)

Now we first compute the open loop transfer function associated with each transport equation. Applying the Laplace transform with respect to t, we obtain the equations

$$sy_i(s,x) + v\,\partial_x y_i(s,x) = 0, \quad i \in \{1,2\},$$

whose solution is $y_i(s,x) = y_i(s,0)e^{-sx/v}$. To find the transfer function we must compute the ratio, after applying the Laplace transform, between the input and output of the system. In our case, what enters the system is $y_i(t,0)$ and what exits is $y_i(t,1)$. Hence the transfer function is given by

$$f_0(s) = \frac{\mathcal{L}(y_i(t,1))}{\mathcal{L}(y_i(t,0))} = \frac{y_i(s,1)}{y_i(s,0)} = \frac{y_i(s,0)e^{-s/v}}{y_i(s,0)} = e^{-s\tau}, \qquad \tau = \frac{1}{v}.$$

We want to use the previous argument for the transport equation in our coupled system. To do so, denote by $Z(s)$ the intermediate signal that comes out of the summing junction $\oplus$. Also, denote by $J(s)$ and $H(s)$ the transfer equations of the transport equations with solution y_1 and y_2 respectively. Then we know that $Y(s) = J(s)Z(s)$ and $Z(s) = U(s) + Y(s)H(s)$. Hence

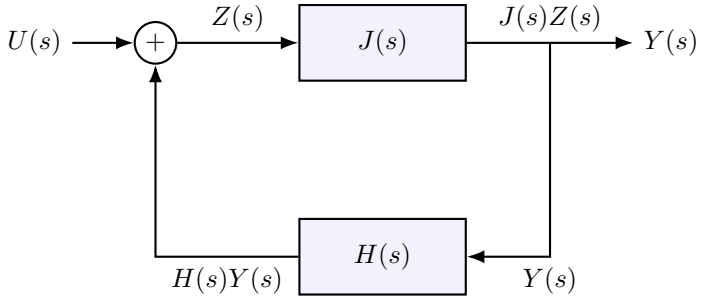


Figure 3.2: Block diagram: The system transfer function is given by $G(s) = Y(s)/U(s) = J(s)/(1 - H(s)J(s))$.

$$\begin{aligned} Y(s) = J(s)Z(s) = J(s)(U(s) + Y(s)H(s)) &\iff Y(s)(1 - H(s)J(s)) = J(s)U(s) \\ &\iff \frac{Y(s)}{U(s)} = \frac{J(s)}{1 - H(s)J(s)}. \end{aligned}$$

Since both equations are transport equations with the same transfer function, $J(s) = H(s) = e^{-s\tau}$, the system transfer equation is given by

$$G(s) = \frac{Y(s)}{U(s)} = \frac{e^{-s\tau}}{1 - e^{-2s\tau}} = \frac{e^{s\tau}}{e^{2s\tau} - 1}. \tag{3.2}$$

That is, the poles of the system are given by the characteristic equation

$$e^{2s\tau} - 1 = 0.$$

This open-loop system is not asymptotically stable, since its poles lie on the imaginary axis, in particular, 0 is a pole. The solutions of the previous equation are given by

$$s_n = \frac{2\pi i n}{\tau}, \qquad n \in \mathbb{Z}.$$

Although the system looks simple, we will show that it cannot be stabilized using a control proportional to the observation, because the control and the observation are anti-located. Consider the control given by

$$U(t) = -2k_p Y(t), \tag{3.3}$$

where $k_p \neq 0$ is a parameter. Substituting into the expression of the transfer function, we obtain

$$\frac{Y(s)}{U(s)} = \frac{-1}{2k_p} = \frac{e^{s\tau}}{e^{2s\tau} - 1},$$

from which we deduce

$$e^{2s\tau} + 2k_p e^{s\tau} - 1 = 0.$$

The solutions of this quadratic equation are given by

$$e^{s_{1,2}\tau} = -k_p \pm \sqrt{1 + k_p^2},$$

so that, writing $s = \sigma + i\omega$ and using $z\overline{z} = |z|^2$ for a complex number z, we get

$$e^{\sigma_{1,2}\tau} = \left| -k_p \pm \sqrt{1 + k_p^2} \right|,$$

which gives two values for the real part, the first one is

$$\sigma_1 = v \ln\left(\left| -k_p + \sqrt{1 + k_p^2} \right|\right) = v \ln\left(-|k_p| + \sqrt{1 + k_p^2}\right) > 0,$$

where we used $\sqrt{1 + k_p^2} > |k_p|$. The other real part is given by

$$\sigma_2 = v \ln\left(\left| -k_p - \sqrt{1 + k_p^2} \right|\right) = v \ln\left(|k_p| + \sqrt{1 + k_p^2}\right) < 0.$$

Note that the phase ω remains free, which tells us that there are infinitely many poles $\sigma + i\omega$ lying on two vertical lines with real parts σ_1 and σ_2 respectively. Hence system (3.2) cannot be stabilized with a static control of the form (3.3).

We now show that it *is* possible to stabilize this system by considering a dynamic control involving a delayed output feedback. Note first that, by (3.2),

$$G(s) = \frac{Y(s)}{U(s)} = \frac{e^{s\tau}}{e^{2s\tau} - 1} \iff Y(s) - Y(s)e^{-2s\tau} = U(s)e^{-s\tau}.$$

Applying the inverse Laplace transform, we obtain

$$Y(t) - Y(t - 2\tau) = U(t - \tau).$$

To stabilize the system, we consider the following control,

$$U(t) = -2k_1 Y(t) - k_2 Y(t - \tau), \tag{3.4}$$

where k_1, k_2 are control parameters to be determined. With this feedback law, we obtain the following delay-difference equation for the closed loop system,

$$Y(t) + 2k_1 Y(t-\tau) + (k_2 - 1)Y(t-2\tau) = 0.$$

Applying the Laplace transform again and multiplying by $e^{2s\tau}$, we obtain

$$Y(s)\left[e^{2s\tau} + 2k_1 e^{s\tau} + (k_2 - 1)\right] = 0,$$

from which it follows that if we choose k_1 and k_2 so that the solutions $w = e^{s\tau}$ of the previous equation lie in the unit disc, we obtain stabilization of the system.

As mentioned before, this must be taken with caution, since boundary feedback stabilization of hyperbolic systems with delayed controls is sensitive to perturbations. We now show this issue. Consider the system

$$\begin{cases} \partial_t \hat{y}_2(t,x) + v\partial_x \hat{y}_2(t,x) = 0, \\ \hat{y}_2(t,0) = y_1(t,1), \\ U(t) = -2k_1 y_1(t,1) - k_2 \hat{y}_2(t,1), \end{cases} \tag{3.5}$$

which is a time-domain representation of the dynamic feedback law (3.4). Here the first two equations represent a transport equation with transport velocity $v = 1/\tau$, equivalent to the delay τ of the feedback law (3.4).

Indeed, to see this it suffices to observe that the equation for $\hat{y}_2$ adds an extra level of delay to the system, since it solves the transport equation whose transfer function is $f_0(s) = e^{-s\tau}$. Hence $\hat{y}_2$ induces a delay of $-\tau$, which lets us say that the law $U(t) = -2k_1 y_1(t,1) - k_2 \hat{y}_2(t,1)$ is equivalent to the previous law.

With this definition, the closed-loop control problem given by (3.1) and (3.5) is represented as

$$\begin{cases} \partial_t y_1(t,x) + v\partial_x y_1(t,x) = 0, \\ \partial_t y_2(t,x) + v\partial_x y_2(t,x) = 0, \\ \partial_t \hat{y}_2(t,x) + v\partial_x \hat{y}_2(t,x) = 0, \\ \begin{pmatrix} y_1(t,0) \\ y_2(t,0) \\ \hat{y}_2(t,0) \end{pmatrix} = \begin{pmatrix} -2k_1 & 1 & -k_2 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} y_1(t,1) \\ y_2(t,1) \\ \hat{y}_2(t,1) \end{pmatrix} \end{cases} \tag{3.6}$$

where we denote by $\mathbf{K}$ the feedback matrix.

After a few computations, diagonalizing the 2×2 lower block first and then closing the loop through the first row of $\mathbf{K}$, one can show that

$$\overline{\rho}(\mathbf{K}) = |k_1| + \sqrt{1 + k_1^2} + |k_2| \geq 1, \quad \forall (k_1, k_2) \in \mathbb{R}^2,$$

where

$$\overline{\rho}(\mathbf{K}) := \max\{\rho(\mathrm{diag}\{e^{-i\theta_1}, e^{-i\theta_2}, e^{-i\theta_3}\}\mathbf{K}); (\theta_1, \theta_2, \theta_3)^t \in \mathbb{R}^3\},$$

and $\rho(M)$ is the spectral radius of a matrix M. It is known that a necessary and sufficient condition for robust stability under perturbations in the nominal velocity is $\overline{\rho}(\mathbf{K}) < 1$ [BC16, Chapter 3].

Even though the closed-loop system may be stable, i.e. having chosen k_1, k_2 so that the poles of the characteristic equation lie in the left half of the complex plane, stability can be

lost due to perturbations of the characteristic velocities of the three transport equations. For instance, if we take velocities $v+\varepsilon_i$ for the first two transport equations corresponding to the plant in (3.6), the system can become unstable with poles moving into the right half of the complex plane for arbitrarily small perturbations. The reason for this is the difference in characteristic velocities between the first two plant equations and the last (control) equation that we added.

One might then think that the feedback law (3.4) is useless in practice. This chapter sets out to prove the opposite. The main contribution is to prove that, even with the choice of control above, the robustness of the output-feedback stabilization against delay perturbations can be recovered as soon as a small diffusion parameter is present in the system. The analysis shows that the stabilizing effect of adding an arbitrarily small diffusion parameter is far from an obvious tool. In fact, diffusion strengthens the robustness of stability for the 2×2 system given by

$$\begin{cases} \partial_t y_1(t,x) + (v+\varepsilon_1)\partial_x y_1(t,x) = 0, \\ \partial_t y_2(t,x) + (v+\varepsilon_2)\partial_x y_2(t,x) = 0, \end{cases}$$

while, on the other hand, it can destroy the stability of similarly simple systems, as will be seen in Chapter 4.

In particular, the dead beat control case occurs when we take $k_1 = 0$ and $k_2 = 1$ leaving us with the characteristic equation $e^{2s\tau} = 0$, meaning that every pole of the system is 0. For instance, those poles have real part tending to $-\infty$. But for such a system we have

$$\overline{\rho}(\mathbf{K}) = \sqrt{2},$$

which shows a lack of robustness of the control with respect to inaccuracies in the delay.

3.3 The Open-Loop Control System with Diffusion

The system that we study is built by interconnecting two identical transport equations perturbed by a diffusive term that can be interpreted as viscosity in the case of a fluid. For simplicity we assume $v = 1$. The system in question is

$$\begin{cases} \partial_t y_1(t,x) + \partial_x y_1(t,x) - \eta\partial^2_{xx} y_1(t,x) = 0, \\ \partial_t y_2(t,x) + \partial_x y_2(t,x) - \eta\partial^2_{xx} y_2(t,x) = 0, \\ y_1(t,0) = y_2(t,1) + U(t), \\ y_2(t,0) = y_1(t,1), \\ \partial_x y_1(t,1) = \partial_x y_2(t,1) = 0, \\ Y(t) = y_1(t,1). \end{cases} \tag{3.7}$$

As before, $U(t)$ is the control, $Y(t)$ the observation, and $\eta > 0$ the viscosity coefficient. The strategy is similar: we apply the Laplace transform in time, $\mathcal{L}[y_i(\cdot,x)](s,x)$, to obtain the system

$$\begin{cases} sy_1(s,x) + \partial_x y_1(s,x) - \eta\partial_{xx}^2 y_1(s,x) = 0, \\ sy_2(s,x) + \partial_x y_2(s,x) - \eta\partial_{xx}^2 y_2(s,x) = 0, \\ y_1(s,0) = y_2(s,1) + U(s), \\ y_2(s,0) = y_1(s,1), \\ \partial_x y_1(s,1) = \partial_x y_2(s,1) = 0, \\ Y(s) = y_1(s,1). \end{cases} \tag{3.8}$$

For the two perturbed transport equations we have solutions of the form

$$y_i(s,x) = A_i(s)e^{\lambda_1(s)x} + B_i(s)e^{\lambda_2(s)x}, \quad i = 1,2,$$

where $\lambda_1(s), \lambda_2(s)$ are the roots of the polynomial

$$\eta\lambda^2 - \lambda - s = 0,$$

so that

$$\lambda_1(s) = \frac{1+\sqrt{1+4\eta s}}{2\eta}, \qquad \lambda_2(s) = \frac{1-\sqrt{1+4\eta s}}{2\eta}. \tag{3.9}$$

Here $\sqrt{\cdot}$ denotes the principal branch of the square root, which is well defined except when $1 + 4\eta s \in \mathbb{R}_-$. This last case implies that $s \in \mathbb{R}$ and $s \leq -1/4\eta$, so it diverges to $-\infty$ as $\eta \to 0$.

We now find the constants $A_i(s), B_i(s)$. From the boundary conditions, evaluated at $x = 0$ and $x = 1$, we obtain the equations

$$\begin{cases} A_i(s) + B_i(s) = y_i(s,0), \\ \lambda_1(s)A_i(s)e^{\lambda_1(s)} + \lambda_2(s)B_i(s)e^{\lambda_2(s)} = 0, \\ y_i(s,1) = A_i(s)e^{\lambda_1(s)} + B_i(s)e^{\lambda_2(s)}. \end{cases} \tag{3.10}$$

Solving,

$$A_i(s) = -\frac{\lambda_2(s)B_i(s)e^{\lambda_2(s)}}{\lambda_1(s)e^{\lambda_1(s)}},$$

we obtain

$$f_\eta(s) = \frac{y_i(s,1)}{y_i(s,0)} = \frac{\left(1 - \frac{\lambda_2(s)}{\lambda_1(s)}e^{\lambda_2(s)}\right)}{-\frac{\lambda_2(s)e^{\lambda_2(s)}}{\lambda_1(s)e^{\lambda_1(s)}} + 1} = \frac{\frac{(\lambda_1(s)-\lambda_2(s))e^{\lambda_2(s)}}{\lambda_1(s)}}{\frac{\lambda_1(s)e^{\lambda_1(s)}-\lambda_2(s)e^{\lambda_2(s)}}{\lambda_1(s)e^{\lambda_1(s)}}} = \frac{\lambda_1(s) - \lambda_2(s)}{\lambda_1(s)e^{-\lambda_2(s)} - \lambda_2(s)e^{-\lambda_1(s)}}.$$

Here we use η as a subscript to emphasize the dependence of the transfer function on the viscosity parameter η. In particular, when $\eta = 0$ we recover $f_0(s) = e^{-s\tau}$.

Now that we know the transfer function, we can rewrite the control system (3.8) as

$$\begin{cases} y_1(s,1) = f_\eta(s)y_1(s,0), \\ y_2(s,1) = f_\eta(s)y_2(s,0), \\ y_1(s,0) = y_2(s,1) + U(s), \\ y_2(s,0) = y_1(s,1), \\ Y(s) = y_1(s,1). \end{cases} \tag{3.11}$$

Eliminating $y_i(s,0), y_i(s,1)$ for $i = 1, 2$, we find that the overall transfer function between input and output for (3.7) is

$$G(s) = \frac{Y(s)}{U(s)} = \frac{f_\eta(s)}{1 - f_\eta^2(s)}.$$

Indeed, using the system above with its transfer functions,

$$\begin{aligned} G(s) = \frac{Y(s)}{U(s)} &= \frac{y_1(s,1)}{y_1(s,0) - y_2(s,1)} \\ &= \frac{f_\eta(s) y_1(s,0)}{y_1(s,0) - f_\eta(s) y_2(s,0)} \\ &= \frac{f_\eta(s) y_1(s,0)}{y_1(s,0) - f_\eta^2(s) y_1(s,0)} \\ &= \frac{f_\eta(s)}{1 - f_\eta^2(s)}, \end{aligned}$$

where we used that the two systems are connected via $y_1(s,1) = y_2(s,0)$.

We want to study the poles of the transfer function, i.e., the roots of the characteristic equation

$$f_\eta^2(s) - 1 = 0.$$

In particular, one notes that $f_\eta(0) = 1$ for every $\eta \neq 0$, meaning there is a pole at the origin. Hence, as in the inviscid case, the open-loop system (3.8) is not asymptotically stable for any value of the viscosity η.

In the next section we show that an output-feedback stabilization result can be obtained for this system when the viscosity is unknown, small, and almost negligible.

3.4 Output Feedback Stabilization of the Viscous System

Here we consider system (3.7), originally open-loop, but now closed via the feedback

$$U(t) = -Y(t-1).$$

This corresponds to the special case of control (3.4) with $k_1 = 0$ and $k_2 = 1$, together with $\tau = 1/v = 1$. In this case, the system is stable, but the stability is not robust with respect to perturbations of the propagation speed v. We will see that including the viscous terms yields stability, and in the next section we will see that, in this case, stability is robust with respect to the perturbation.

In the frequency domain, system (3.1)–(3.5) together with the feedback above gives us the system

$$\begin{cases} y_1(s,1) = f_\eta(s) y_1(s,0), \\ y_2(s,1) = f_\eta(s) y_2(s,0), \\ \tilde{y}_2(s,1) = e^{-s} \tilde{y}_2(s,0), \\ y_1(s,0) = y_2(s,1) - \tilde{y}_2(s,1), \\ y_2(s,0) = y_1(s,1), \\ \tilde{y}_2(s,0) = y_1(s,1), \end{cases} \tag{3.12}$$

where, for the feedback, we note that, since

$$\mathcal{L}(U(t))(s) = U(s) = \mathcal{L}(-Y(t-1))(s) = -e^{-s} Y(s) = -e^{-s} y_1(s,1),$$

we have

$$y_1(s,0) = y_2(s,1) + \mathcal{L}(U(t))(s) = y_2(s,1) - e^{-s}y_1(s,1).$$

We claim that this system admits the following characteristic equation,

$$f_\eta(s)^2 - f_\eta(s)e^{-s} - 1 = 0. \tag{3.13}$$

Indeed, on the one hand,

$$\begin{aligned}\frac{Y(s)}{U(s)} &= \frac{\mathcal{L}[y_1(t,1)](s)}{\mathcal{L}[y_1(t,0) - y_2(t,1)](s)} \\ &= \frac{y_1(s,1)}{y_1(s,0) - y_2(s,1)} \\ &= \frac{f_\eta(s)y_1(s,0)}{y_1(s,0) - f_\eta(s)y_2(s,0)} \\ &= \frac{f_\eta y_1(s,0)}{y_1(s,0) - f_\eta^2(s)y_1(s,0)},\end{aligned}$$

where we used system (3.12). On the other hand,

$$\frac{Y(s)}{U(s)} = \frac{f_\eta(s)y_1(s,0)}{-e^{-s}f_\eta(s)y_1(s,0)} = \frac{1}{-e^{-s}}.$$

That is,

$$\frac{f_\eta(s)}{1 - f_\eta^2(s)} = \frac{1}{-e^{-s}} \iff f_\eta^2(s) - e^{-s}f_\eta(s) - 1 = 0.$$

Now, to prove the stability of this closed-loop system, we define, for viscosity $\eta > 0$, the spectrum $\mathcal{S}_\eta$ of the closed-loop system as the set of poles of the characteristic equation, i.e., the solutions of

$$\mathcal{S}_\eta = \{s : f_\eta^2(s) - f_\eta(s)e^{-s} - 1 = 0\}.$$

Along the same lines, we define the maximal spectral abscissa as the supremum of the real parts of the spectrum,

$$\sigma_\eta = \sup\{\Re(s) : s \in \mathcal{S}_\eta\}.$$

It should be noted that $S_\eta = \emptyset$ means that there are no solutions for the characteristic equation (3.13). Hence there's no pole for the system (3.12), giving the stability for the system which gives $\sigma_\eta = -\infty$ in terms of the maximal spectral abscissa.

The stability of system (3.12) with $\eta = 0$ follows from the spectral mapping theorem [EN00, Cor IV.3.12].

Theorem 3.1 ([EN00])**.** *Let $(T(t))_{t\geq 0}$ be a semigroup with generator $(A, D(A))$ on the Banach space X. Suppose that $(T(t))_{t\geq 0}$ is eventually norm-continuous, i.e, there exists $t_0 \geq 0$ such that the map $t \mapsto T(t)$ is continous for every $t \geq t_0$. The spectral mapping theorem*

$$e^{t\sigma(A)} = \sigma(T(t)) \setminus \{0\}, \quad t \geq 0,$$

and the spectral bound equal growth bound condition

$$s(A) = \omega_0(T),$$

where the spectral abscissa and the growth bound are defined as

$$s(A) := \sup\{\Re\lambda : \lambda \in \sigma(A)\}, \quad \omega_0(T) := \{\inf \omega \in \mathbb{R} : \exists M \geq 1 : \|T(t)\| \leq Me^{\omega t}, \forall t \geq 0\},$$

hold for the following classes of C_0-semigroups,

1. *Eventually compact semigroups,*
2. *Eventually differentiable semigroups,*
3. *Analytic semigroups,*
4. *Uniformly continuous semigroups.*

In particular, under the hypotheses of the Spectral Mapping Theorem, this closed-loop semigroup is exponentially stable if and only if $\sigma_\eta < 0$. So if we take $\eta = 0$, we have $f_0(s) = e^{-s}$ and the characteristic equation (3.13) becomes

$$e^{-2s} - e^{-s} \cdot e^{-s} - 1 = -1 \neq 0, \quad \forall s \in \mathbb{C}.$$

This means that $\mathcal{S}_0 = \emptyset$, so $\sigma_0 = -\infty$, which means that system (3.12) converges to 0 in finite time. This gives an exponential stability result since the system satisfies the spectral mapping property.

For the case $\eta \neq 0$, we refer to Section 4.4. It is not known if the system satisfies the spectral mapping property, so here we seek for a different notion of stability which may differ from the exponential one.

The first result of this chapter is the following.

Theorem 3.2 (Theorem 1, [BCH23])**.** *For every $\delta > 0$, there exists $\eta_1 > 0$ such that, for all $\eta \in (0, \eta_1)$, the maximal spectral abscissa satisfies*

$$\sigma_\eta \leq -\ln(2) + \delta.$$

Remark 3.3. Theorem 3.2 says that, once $\eta > 0$ appears attached to a diffusion term, the system becomes stable under the delayed feedback choice for $U(t)$. This is the first step to study the robust stabilization of (3.12) in a similar fashion as in (3.6).

It is important to note that Theorem 3.2 only asserts an upper bound for the spectrum where $\ln(2)$ appears. This is obtained via numerical experiments, and with the tools of this chapter, the authors couldn't obtain a lower bound, so the next conjecture is a natural question.

Conjecture 1. The bound $\ln(2)$ in the decay rate is optimal. More precisely, for every $\delta > 0$ and $\eta_1 > 0$, there exists $\eta \in (0, \eta_1)$ such that the maximal spectral abscissa satisfies

$$-\ln(2) - \delta \leq \sigma_\eta \leq -\ln(2) + \delta.$$

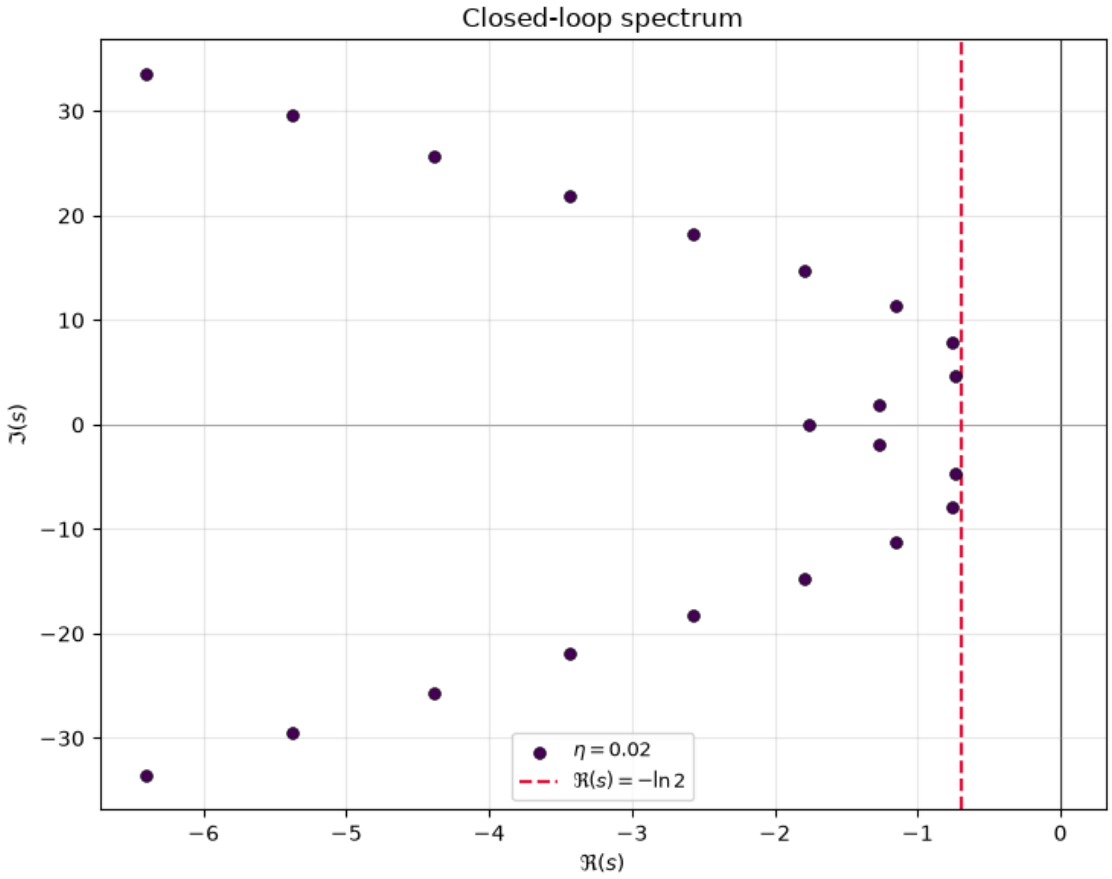


Figure 3.3: Closed-loop spectrum of (3.12) with $\eta = 0.02$

Remark 3.4 (Loss of continuity of the spectral abscissa)**.** The conjecture above is verified numerically when observing the spectrum. This implies a loss of continuity, in the sense that any decay rate is attainable when $\eta = 0$. However, a bound of $\ln(2)$ exists as soon as $\eta > 0$, even for arbitrarily small η.

Before proving the theorem, we define the complex variable

$$z = \sqrt{1 + 4\eta s},$$

where $\sqrt{\cdot}$ is the principal branch of the square root. The functions $\lambda_{1,2}$ become

$$\lambda_1 = \frac{(1+z)}{2\eta}, \qquad \lambda_2 = \frac{(1-z)}{2\eta}.$$

Then, defining the function

$$X_\eta(z) = \frac{1+z}{2} e^{-(1-z)/2\eta} - \frac{1-z}{2} e^{-(1+z)/2\eta},$$

and using the characteristic equation, we obtain, in terms of this new function,

$$X_\eta^2(z) + z e^{-(z^2-1)/4\eta} X_\eta(z) - z^2 = 0. \tag{3.14}$$

Indeed, this follows by noting that $s = \frac{z^2-1}{4\eta}$ and that $\lambda_1(s) - \lambda_2(s) = \frac{z}{\eta}$. Hence

$$0 = f_\eta(s)^2 - e^{-s} f_\eta(s) - 1 = \frac{z^2}{X_\eta^2} - e^{-s} \frac{z}{X_\eta} - 1,$$

from which the equation follows. Note from this equation that $X_\eta(0) = 0$. Moreover, $z = 0$ is a root of (3.14) for every $\eta > 0$. This implies that $s = -1/4\eta \in \mathcal{S}_\eta$ is a pole of the system, i.e., a solution of the characteristic equation. This pole satisfies $s \to -\infty$ as $\eta \to 0^+$, so the pole is stable.

Denote by $\mathcal{Z}_\eta$ the set of non-zero roots of (3.14). Using the definition of $X_\eta(z)$ and solving the quadratic equation in X_η, every $z \in \mathcal{Z}_\eta$ satisfies

$$\frac{1+z}{z}e^{-(1-z)/2\eta} - \frac{1-z}{z}e^{-(1+z)/2\eta} = -e^{-(z^2-1)/4\eta} \pm \sqrt{e^{-(z^2-1)/2\eta} + 4}. \tag{3.15}$$

Now consider a sequence

$$(\eta_n)_{n\in\mathbb{N}} : \quad 0 < \eta_n \in \mathbb{R},\ \forall n \in \mathbb{N}, \qquad \lim_{n\to\infty} \eta_n = 0^+,$$

and an associated sequence

$$(s_n)_{n\in\mathbb{N}} : \quad s_n \in \mathcal{S}_{\eta_n},\ \forall n \in \mathbb{N}.$$

To prove the theorem, we study the accumulation points of $(s_n)_{n\in\mathbb{N}}$. By definition, $\overline{s}$ is an accumulation point of $(s_n)_{n\in\mathbb{N}}$ if there is a subsequence $(s_{n_k})_{k\in\mathbb{N}}$ such that $s_{n_k} \to \overline{s}$. We abuse notation and write $(s_n)_{n\in\mathbb{N}}$ even when working with the subsequence.

The proof of the theorem reduces to proving two technical lemmas.

Lemma 3.5 (Lemma 1, [BCH23])**.** *Let $(\eta_n)_{n\in\mathbb{N}}$ be a sequence as above and let $(z_n)_{n\in\mathbb{N}}$ be the associated sequence of elements of $\mathcal{Z}_{\eta_n}$. Let $\overline{z}$ be an accumulation point of $(z_n)_{n\in\mathbb{N}}$. Then*

1. *$\Re(\overline{z}^2) \le 1$. Precisely, $\Re(\overline{z}^2) \in [-\infty, 1]$.*
2. *If $\Re(\overline{z}^2) = 1$, then $(\Re(\overline{z}))^2 = 1$ and $\Im(\overline{z}) = 0$.*

Proof. 1. For the first part, suppose by contradiction that $\Re(\overline{z}^2) \in (1, +\infty]$. Then $e^{-(\Re(z_n^2)-1)/4\eta_n} \to 0$ and it follows that the right-hand side of (3.15) converges to ± 2. We write $a_n = \Re(z_n)$ and $b_n = \Im(z_n)$, so that the equation implies

$$\frac{1}{2}\left(\left(\frac{a_n - ib_n}{a_n^2+b_n^2} + 1\right)e^{-(1-a_n-ib_n)/2\eta_n} - \left(\frac{a_n - ib_n}{a_n^2+b_n^2} - 1\right)e^{-(1+a_n+ib_n)/2\eta_n}\right) \to \pm 1. \tag{3.16}$$

Since $\Re(\overline{z}^2) \in (1, \infty]$, there exists $c > 0$ such that $a_n^2 > c + 1 + b_n^2$ for n large enough. In particular, $|\Re(\overline{z})| > 1$. Consider then two cases: $\Re(\overline{z}) > 1$ and $\Re(\overline{z}) < -1$.

Case 1: $\Re(\overline{z}) > 1$. For n large enough,

$$\Re\left(\frac{a_n - ib_n}{a_n^2+b_n^2} + 1\right) \ge 1, \qquad e^{(a_n-1)/2\eta_n} \to +\infty.$$

Hence

$$\left|\frac{1}{2}\left(\frac{a_n - ib_n}{a_n^2+b_n^2} + 1\right)e^{(a_n-1)/2\eta_n}\right| \to +\infty,$$

while

$$\frac{1}{2}\left(\frac{a_n - ib_n}{a_n^2+b_n^2}\right)e^{(-a_n-1)/2\eta_n}e^{-ib_n/2\eta_n} \to 0.$$

This implies

$$\left|\frac{1}{2}\left(\frac{a_n - ib_n}{a_n^2+b_n^2} + 1\right)e^{(a_n-1)/2\eta_n}e^{ib_n/2\eta_n} - \frac{1}{2}\left(\frac{a_n - ib_n}{a_n^2+b_n^2} - 1\right)e^{-(1+a_n+ib_n)/2\eta_n}\right| \to +\infty,$$

which contradicts (3.16).

Case 2: we have $\Re(\overline{z}) < -1$. By an analogous argument,

$$e^{(-a_n-1)/2\eta_n} \to +\infty, \qquad \left|\left(\frac{a_n - ib_n}{a_n^2+b_n^2}\right) e^{(-a_n-1)/2\eta_n} e^{-ib_n/2\eta_n}\right| \to +\infty,$$

while

$$\left|\frac{1}{2}\left(\frac{a_n - ib_n}{a_n^2+b_n^2}+1\right) e^{(a_n-1)/2\eta_n} e^{ib_n/2\eta_n}\right| \to 0.$$

This again contradicts (3.16), proving the first part.

2. For the second part, suppose $\Re(\overline{z}^2) = 1$. We have

$$\Re(\overline{z}^2) = (\Re(\overline{z}))^2 - (\Im(\overline{z}))^2.$$

From these two expressions we conclude that either $|\Re(\overline{z})| = 1$ and $\Im(\overline{z}) = 0$, or $|\Re(\overline{z})| > 1$.

We prove by contradiction that $\Re(\overline{z}) > 1$ is impossible (and, by symmetry, the same argument applies to $\Re(\overline{z}) < -1$). Writing again $a_n = \Re(z_n)$ and $b_n = \Im(z_n)$, and multiplying both sides of (3.15) by $e^{(1-a_n)/4\eta_n}$, we get

$$\left(\frac{a_n - ib_n}{a_n^2+b_n^2}+1\right) e^{(a_n-1)/4\eta_n + ib_n/2\eta_n} - \left(\frac{a_n - ib_n}{a_n^2+b_n^2}-1\right) e^{-(1+3a_n+2ib_n)/4\eta_n} \tag{3.17}$$

$$= -e^{(2-a_n^2+b_n^2-a_n-2ia_nb_n)/3\eta_n} \pm \sqrt{e^{(2-a_n^2+b_n^2-a_n-2ia_nb_n)/2\eta_n} + 4e^{(1-a_n)/2\eta_n}}. \tag{3.18}$$

Now, since $1 - a_n \to 1 - \Re(\overline{z}) < 0$, we have, using $\Re(\overline{z}^2) = 1$, that the right-hand side of (3.17) converges to 0, i.e.,

$$\left(\frac{a_n - ib_n}{a_n^2+b_n^2}+1\right) e^{(a_n-1)/4\eta_n + ib_n/2\eta_n} - \left(\frac{a_n - ib_n}{a_n^2+b_n^2}-1\right) e^{-(1+3a_n+2ib_n)/4\eta_n} \to 0.$$

Then, since $e^{-(1+3a_n+ib_n)/4\eta_n} \to 0$, we would need

$$\left|\left(\frac{a_n - ib_n}{a_n^2+b_n^2}+1\right)\right| e^{(a_n-1)/4\eta_n} \to 0,$$

but this is impossible, since $1 - a_n \to 1 - \Re(\overline{z}) < 0$ shows that $a_n > 0$ for n large enough, which implies

$$\left|\left(\frac{a_n - ib_n}{a_n^2+b_n^2}+1\right)\right| e^{(a_n-1)/4\eta_n} \geq e^{(a_n-1)/4\eta_n}.$$

But since $1 - a_n \to 1 - \Re(\overline{z}) < 0$, $e^{(a_n-1)/4\eta} \to +\infty$, which contradicts the last two equations and concludes the proof of the lemma.

□

Lemma 3.6 (Lemma 2, [BCH23]). *Let $(\eta_n)_{n\in\mathbb{N}}$ be a sequence as before and $(s_n)_{n\in\mathbb{N}}$ an associated sequence. Let $\overline{s}$ be an accumulation point of $(s_n)_{n\in\mathbb{N}}$. Then*

$$\Re(\overline{s}) \leq -\ln(2), \quad \textit{precisely, } \Re(\overline{s}) \in [-\infty, -\ln(2)].$$

Proof. Let $\overline{s}$ be the accumulation point above. We restrict to the subsequence $(s_n)_{n\in\mathbb{N}}$ that converges to $\overline{s}$. Consider the sequence $(z_n)_{n\in\mathbb{N}}$ with

$$z_n = \sqrt{1+4\eta_n s_n}, \quad \forall n \in \mathbb{N}.$$

By definition, $z_n \in \mathcal{Z}_{\eta_n}$ for all $n \in \mathbb{N}$. Let $\overline{z}$ be an accumulation point of $(z_n)_{n\in\mathbb{N}}$, and again restrict to a subsequence $(z_n)_{n\in\mathbb{N}}$ that converges to $\overline{z}$. Since it is a subsequence, we still have $s_n \to \overline{s}$. By Theorem 3.5, we necessarily have $\Re(\overline{z}^2) \in [-\infty, 1]$.

We first consider the case where $\Re(\overline{z}^2)$ is strictly less than 1. In this case, by the definition of $(s_n)_{n\in\mathbb{N}}$,

$$\Re(\overline{s}) = \lim_{n\to\infty} \Re(s_n) = \lim_{n\to\infty} \frac{\Re(z_n^2)-1}{4\eta_n} = \frac{\lim_{n\to\infty}\Re(z_n^2)-1}{\lim_{n\to\infty} 4\eta_n} = \frac{\Re(\overline{z}^2)-1}{\lim_{n\to\infty}4\eta_n}.$$

Since $\lim_{n\to\infty} 4\eta_n = 0^+$ by definition of $(\eta_n)_{n\in\mathbb{N}}$, we get $\Re(\overline{s}) = -\infty$.

We now consider the second case, where $\Re(\overline{z}^2) = 1$. By Theorem 3.5, $\Re(\overline{z}) = \pm 1$ and $\Im(\overline{z}) = 0$, so $\overline{z} = \pm 1$. We treat the case $\overline{z} = 1$, since the other case follows the same argument. We use the notation

$$y_n = z_n - 1, \quad a_n = \Re(y_n), \quad b_n = \Im(y_n).$$

With this notation,

$$\Re(s_n) = \frac{\Re(z_n^2)-1}{4\eta_n} = \frac{\Re((y_n+1)^2)-1}{4\eta_n} = \frac{a_n^2-b_n^2}{4\eta_n} + \frac{a_n}{2\eta_n}.$$

Here the identification of accumulation points of $(\Re(s_n))_{n\in\mathbb{N}}$ is induced by the limit $\Re(z_n) \to 1$, which is not an immediate consequence of the identity above, so we treat this case more carefully. Note that, in the new notation,

$$X_\eta = \left(1+\frac{y_n}{2}\right)e^{y_n/2\eta_n} + \frac{y_n}{2}e^{-1/\eta_n - y_n/2\eta_n},$$

so that the characteristic equation (3.14) reads

$$\left(\left(1+\frac{y_n}{2}\right)e^{y_n/2\eta_n} + \frac{y_n}{2}e^{-1/\eta_n-y_n/2\eta_n}\right)^2 \tag{3.19}$$

$$+ (1+y_n)e^{-y_n/2\eta_n - y_n^2/4\eta_n}\left[\left(1+\frac{y_n}{2}\right)e^{y_n/2\eta_n} + \frac{y_n}{2}e^{-1/4\eta_n - y_n/2\eta_n}\right] = (1+y_n)^2. \tag{3.20}$$

Now, since $\eta_n \to 0^+$ and $y_n \to 0$ (because $z_n \to \overline{z}$), we claim this implies

$$\left(1+\frac{y_n}{2}\right)^2 e^{y_n/\eta_n} + o(e^{-1/\eta_n}) + (1+y_n)\left(1+\frac{y_n}{2}\right)e^{-y_n^2/4\eta_n} \tag{3.21}$$

$$+ o(e^{-1/4\eta_n - y_n/\eta_n - y_n^2/4\eta_n}) = (1+y_n)^2. \tag{3.22}$$

Indeed, for the first squared term in (3.19), expanding gives

$$\left(1+\frac{y_n}{2}\right)^2 e^{y_n/\eta_n} + \left(1+\frac{y_n}{2}\right)\frac{y_n}{2}\exp\left(\frac{y_n}{2\eta_n} - \frac{1}{\eta_n} - \frac{y_n}{2\eta_n}\right) + \left(\frac{y_n}{2}\right)^2 e^{-2/\eta_n - y_n/\eta_n}.$$

The first term matches with the first expression in (3.21). It remains to check that the rest of the expression vanishes in the limit when divided by e^{-1/η_n},

$$\lim_{n\to\infty}\left(1+\frac{y_n}{2}\right)\frac{y_n}{2} + \left(\frac{y_n}{2}\right)^2 e^{-1/\eta_n(1+y_n)} = 0 \cdot e^{-1/0^+} = 0,$$

which proves the first part. For the second mixed term on the left hand side of (3.20), the first summand of the expansion is the one we want, so we only need the second summand to be $o(e^{-y_n/\eta_n - 1/4\eta_n})$. Indeed,

$$\lim_{n\to\infty}(1+y_n)\frac{y_n}{2}e^{-y_n^2/4\eta_n} = 0\cdot e^{-1/0^+} = 0,$$

which proves the second part. We next claim that (3.21) implies

$$e^{y_n/\eta_n} + \frac{1+y_n}{1+y_n/2}e^{-y_n^2/4\eta_n} \to 1.$$

Indeed, note that

$$\left(\frac{1+y_n}{1+y_n/2}\right)^2 = e^{y_n/\eta_n} + \frac{o(e^{-1/\eta_n})}{(1+y_n/2)^2} + \frac{1+y_n}{1+y_n/2}e^{-y_n^2/4\eta_n} + \frac{o(e^{-1/4\eta_n - y_n/\eta_n - y_n^2/4\eta_n})}{(1+y_n/2)^2},$$

so that

$$\lim_{n\to\infty} e^{y_n/\eta_n} + \frac{1+y_n}{1+y_n/2}e^{-y_n^2/4\eta_n} = \lim_{n\to\infty}\left(\frac{1+y_n}{1+y_n/2}\right)^2 - \frac{o(e^{-1/\eta_n})}{(1+y_n/2)^2} - \frac{o(e^{-1/4\eta_n - y_n/\eta_n - y_n^2/4\eta_n})}{(1+y_n/2)^2}.$$

On the right-hand side, the first limit is 1, for the other two,

$$\lim_{n\to\infty}\frac{-o(e^{-1/\eta_n})}{e^{-1/\eta_n}}\frac{e^{-1/\eta_n}}{(1+y_n/2)^2} - \frac{o(e^{-1/4\eta_n - y_n/\eta_n - y_n^2/4\eta_n})}{e^{-1/4\eta_n - y_n/\eta_n - y_n^2/4\eta_n}}\cdot\frac{e^{-1/4\eta_n - y_n/\eta_n - y_n^2/4\eta_n}}{(1+y_n/2)^2} = 0,$$

which proves the claim. In terms of the real and imaginary parts a_n, b_n,

$$e^{a_n/\eta_n}e^{ib_n/\eta_n} + \frac{(1+a_n+ib_n)(1+a_n/2-ib_n/2)}{|1+y_n/2|^2}e^{-a_n^2/4\eta_n}e^{b_n^2/4\eta_n}e^{-ia_nb_n/2\eta_n} \to 1.$$

Separating real and imaginary parts,

$$e^{a_n/\eta_n}\cos\left(\frac{b_n}{\eta_n}\right) + e^{-a_n^2/4\eta_n}e^{b_n^2/4\eta_n}. \tag{3.23}$$

$$\cdot\left[\left(1+\frac{3a_n+a_n^2+b_n^2}{2}\right)\cos\left(\frac{a_nb_n}{2\eta_n}\right) + \frac{b_n}{2}\sin\left(\frac{a_nb_n}{2\eta_n}\right)\right]\frac{1}{|1+y_n/2|^2} \to 1, \tag{3.24}$$

$$e^{a_n/\eta_n}\sin\left(\frac{b_n}{\eta_n}\right) + e^{-a_n^2/4\eta_n}e^{b_n^2/4\eta_n}. \tag{3.25}$$

$$\cdot\left[-\left(1+\frac{3a_n+a_n^2+b_n^2}{2}\right)\sin\left(\frac{a_nb_n}{2\eta_n}\right) + \frac{b_n}{2}\cos\left(\frac{a_nb_n}{2\eta_n}\right)\right]\frac{1}{|1+y_n/2|^2} \to 0. \tag{3.26}$$

Let $\overline{a}$ denote an accumulation point of $(a_n/2\eta_n)_{n\in\mathbb{N}}$. We discuss three cases: $\overline{a} = -\infty$, $\overline{a}\in\mathbb{R}$, and $\overline{a} = +\infty$.

In the first case, $a_n \to 0$ and $a_n/2\eta_n \to -\infty$. It follows that

$$\lim_{n\to\infty}\frac{a_n^2}{4\eta_n} + \frac{a_n}{2\eta_n} = \lim_{n\to\infty}\frac{a_n}{2\eta_n}\left(\frac{a_n}{2}+1\right) = \overline{a} = -\infty,$$

hence, since

$$\Re(s_n) = \frac{a_n^2 - b_n^2}{4\eta_n} + \frac{a_n}{2\eta_n},$$

and

$$\lim_{n\to\infty}\frac{a_n^2-b_n^2}{4\eta_n} + \frac{a_n}{2\eta_n} = \lim_{n\to\infty}\frac{a_n}{2\eta_n}\frac{a_n}{2} - \frac{b_n^2}{4\eta_n} + \frac{a_n}{2\eta_n} = 0 - 0 - \infty,$$

we get $\Re(s_n) \to -\infty$. This proves the first case. For the second case, note that $a_n^2/2\eta_n \to 0$ and $a_n b_n/2\eta_n \to 0$, since $a_n \to 0$ and $b_n \to 0$. From the expression for the real part (3.24) we claim that

$$e^{a_n/\eta_n}\cos\left(\frac{b_n}{\eta_n}\right) + e^{b_n^2/4\eta_n}(1+o(1)) \to 1.$$

Since $(a_n^2/2\eta_n), (a_n b_n)/2\eta_n \to 0$, they are $o(1)$. Hence

$$e^{-a_n^2/4\eta_n} \to 1, \quad \left(1 + \frac{3a_n + a_n^2 + b_n^2}{2}\right) \to 1, \quad \cos\left(\frac{a_n b_n}{2\eta_n}\right) \to 1, \quad \sin\left(\frac{a_n b_n}{2\eta_n}\right) \to 0.$$

Therefore, limit (3.24) implies, in our case,

$$e^{a_n/\eta_n}\cos\left(\frac{b_n}{\eta_n}\right) + e^{b_n^2/4\eta_n}(1+o(1)) \to 1. \tag{3.27}$$

Let $\overline{c} \in [0,\infty]$ denote an accumulation point of $(b_n^2/4\eta_n)_{n\in\mathbb{N}}$ and $\overline{\kappa} \in [-1,1]$ an accumulation point of $\left(\cos\left(\frac{b_n}{\eta_n}\right)\right)_{n\in\mathbb{N}}$. Then, up to a diagonal argument so that all sequences considered converge to the accumulation points chosen above, we can pass to the limit in (3.27) and obtain

$$e^{2\overline{a}}\overline{\kappa} + e^{\overline{c}} = 1.$$

Since $e^{\overline{a}}$ is bounded and positive, together with $b_n^2/4\eta_n \geq 0$ for all $n \in \mathbb{N}$, this implies $\overline{c} \in [0,\infty)$ with $e^{\overline{c}} \geq 1$ and $\overline{\kappa} \in [-1,0]$. Indeed, if $\overline{\kappa} = 0$, then $\overline{c} = 0$. Now, if $\overline{\kappa} \neq 0$, we have

$$e^{\overline{a}} = \frac{1 - e^{\overline{c}}}{\overline{\kappa}} > 0.$$

We have two options: if $\overline{\kappa} \in [-1,0)$, then $1 < e^{\overline{c}}$. The second option is that, for $\overline{\kappa} \in (0,1]$, we would need $\overline{c} < 0$, which cannot happen since $\overline{c} \geq 0$.

Hence $e^{\overline{c}} \geq 1$, $\overline{\kappa} \in [-1,0]$ and $\overline{c} \in [0,\infty)$. In particular, b_n^2/η_n is bounded as $\eta_n \to 0^+$, so we can use the imaginary part (3.26) to obtain

$$e^{a_n/\eta_n}\sin\left(\frac{b_n}{\eta_n}\right) + o(1) \to 0,$$

so $\sin\left(\frac{b_n}{\eta_n}\right) \to 0$, meaning the only accumulation point for $\cos(b_n/\eta_n)$ is $\overline{\kappa} = -1$. Hence

$$e^{\overline{c}} = 1 + e^{2\overline{a}}.$$

Finally, considering $e^{-\Re(s_n)}$,

$$e^{-\Re(s_n)} = e^{(b_n^2 - a_n^2)/4\eta_n - a_n/2\eta_n} \to e^{\overline{c}}e^{-\overline{a}} = e^{-\overline{a}} + e^{\overline{a}} = 2\cosh(\overline{a}) \geq 2.$$

It follows that $\Re(\overline{s}) \leq -\ln(2)$, which proves the result in this case.

Finally we study the case $\overline{a} = \infty$. Note that $a_n/2\eta_n \to +\infty$ and

$$\frac{a_n^2}{4\eta_n} = \frac{a_n}{2}\frac{a_n}{2\eta_n} \to 0\cdot\infty = 0, \qquad \frac{b_n^2}{4\eta_n} = \frac{b_n^2}{4}\frac{1}{\eta_n} \to 0\cdot+\infty = 0,$$

hence $e^{(a_n^2 - b_n^2)/4\eta_n - a_n/2\eta_n} \to 0$. We now study the real part (3.24). Multiplying by $e^{(a_n^2-b_n^2)/4\eta_n - a_n/2\eta_n}$, we have

$$e^{(a_n^2-b_n^2)/4\eta_n + a_n/2\eta_n}\cos\left(\frac{b_n}{\eta_n}\right) + e^{-a_n/2\eta_n}\left[\left(1 + \frac{3a_n + a_n^2 + b_n^2}{2}\right)\right]\frac{1}{|1+\frac{y_n}{2}|^2} - e^{(a_n^2-b_n^2)/4\eta_n - a_n/2\eta_n} \to 0.$$

We study the three components of this limit. The third term converges to 0 by the previous argument. For the second term,

$$\lim_{n\to\infty} e^{-a_n/2\eta_n}\left[\left(1+\frac{3a_n+a_n^2+b_n^2}{2}\right)\right]\frac{1}{|1+\frac{y_n}{2}|^2} = e^{-\infty}((1+0)\cdot 1+0) = 0.$$

Finally, since the previous limits vanish, the first term must also converge to 0, i.e.,

$$e^{(a_n^2-b_n^2)/4\eta_n+a_n/2\eta_n}\cos\left(\frac{b_n}{\eta_n}\right)\to 0.$$

Repeating the argument with the imaginary part (3.26), we analogously obtain

$$e^{(a_n^2-b_n^2)/4\eta_n+a_n/2\eta_n}\sin\left(\frac{b_n}{\eta_n}\right)\to 0.$$

Multiplying the two expressions, we necessarily have

$$e^{(a_n^2-b_n^2)/2\eta_n+a_n/\eta_n} = e^{2\Re(s_n)}\to 0.$$

This implies $\Re(s_n)\to-\infty$, which proves the last case of the lemma. □

The proof of Theorem 3.2 now follows quickly.

Proof of Theorem 3.2. Assume, by contradiction, that the conclusion of the theorem is false, i.e., for every $\delta>0$ there is no $\eta_1>0$ such that $\sigma_\eta\le-\ln(2)+\delta$ for all $\eta\in(0,\eta_1)$. In other words, for every $\delta>0$ there exist $\eta_1\in(0,\eta)$ and $s\in\mathcal{S}_{\eta_1}$ such that $\Re(s)>-\ln(2)+\delta$.

It follows that we can construct a sequence of diffusion parameters $(\eta_n)_{n\in\mathbb{N}}$, positive and with $\eta_n\to 0^+$, with an associated sequence $(s_n)_{n\in\mathbb{N}}$ with $s_n\in\mathcal{S}_{\eta_n}$. For this sequence, we have, by the contradiction hypothesis, that $\Re(\bar{s})>-\ln(2)+\delta$, which contradicts Lemma 3.6. □

3.5 Robustness Analysis

We consider the same control system, but we want to study the sensitivity to velocity perturbations. To this end we introduce $\varepsilon>0$, a perturbation of the nominal velocity $v=1$, so that the system becomes

$$\begin{cases}\partial_t y_1(t,x)+(1+\varepsilon)\partial_x y_1(t,x)-\eta\partial_{xx}^2 y_1(t,x)=0,\\ \partial_t y_2(t,x)+(1+\varepsilon)\partial_x y_2(t,x)-\eta\partial_{xx}^2 y_2(t,x)=0,\\ y_1(t,0)=y_2(t,1)+U(t),\\ y_2(t,0)=y_1(t,1),\\ \partial_x y_1(t,1)=\partial_x y_2(t,1)=0,\\ Y(t)=y_1(t,1).\end{cases} \tag{3.28}$$

We again use the feedback $U(t)=-Y(t-1)$. Note that this control depends on the delay $\tau=1$. For simplicity we also assume that the perturbation is the same for both transport equations. This should not be seen as a simplification, since even an arbitrarily small perturbation is able to break stability when $\eta=0$.

In this case, the transfer function becomes

$$f_{\eta,\varepsilon}(s)=\frac{\lambda_1(s)-\lambda_2(s)}{\lambda_1(s)e^{-\lambda_2(s)}-\lambda_2(s)e^{-\lambda_1(s)}},$$

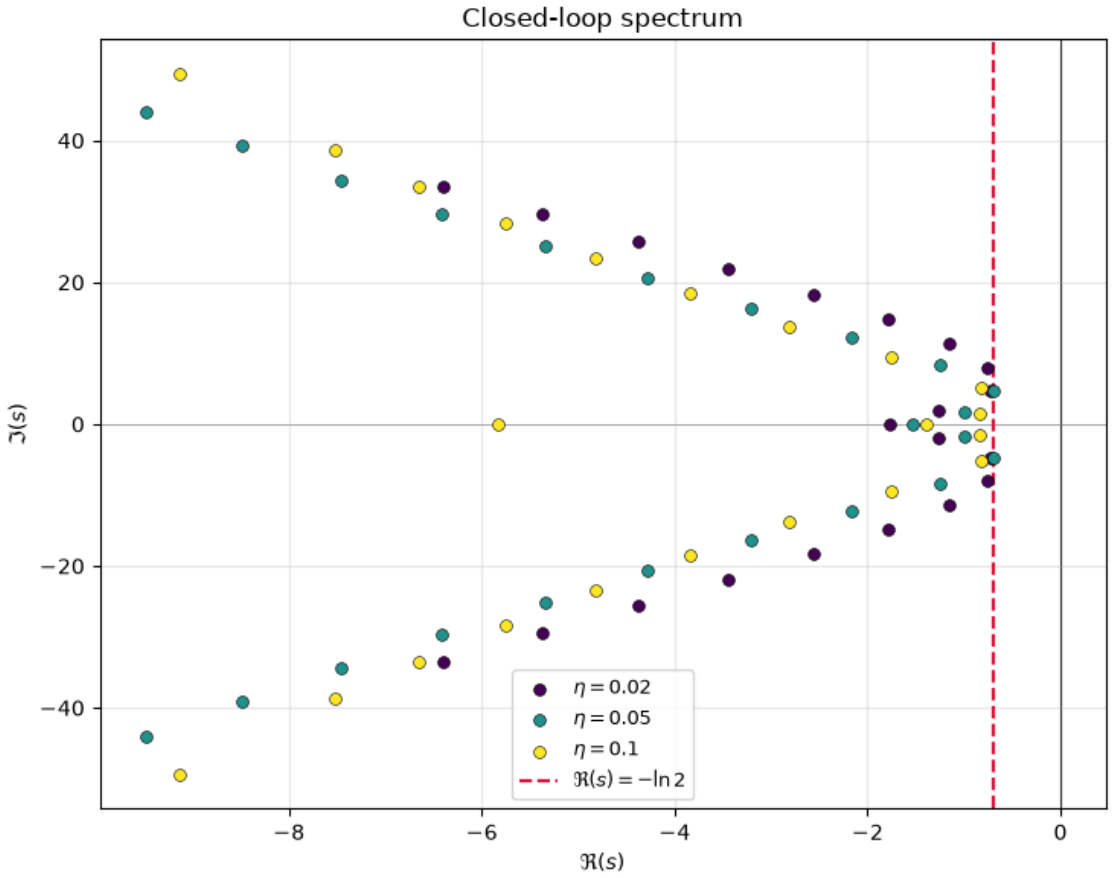


Figure 3.4: Closed-loop spectrum of (3.12) with $\eta = 0.02, 0.05, 0.1$

with functions $\lambda_{1,2}$ that also change to account for the perturbation,

$$\lambda_{1,2}(s) := \frac{(1+\varepsilon) \pm \sqrt{(1+\varepsilon)^2 + 4\eta s}}{2\eta}.$$

With this definition, we have the characteristic equation

$$f_{\eta,\varepsilon}(s)^2 - f_{\eta,\varepsilon}(s)e^{-s} - 1 = 0. \tag{3.29}$$

As before, we define the perturbed spectrum,

$$\mathcal{S}_{\eta,\varepsilon} = \{s \in \mathbb{C} : s \text{ solves } (3.29)\},$$

and the maximal spectral abscissa

$$\sigma_{\eta,\varepsilon} = \sup\{\Re(s) : s \in \mathcal{S}_{\eta,\varepsilon}\}.$$

In addition, note that

$$\mathcal{S}_{\eta,0} = \mathcal{S}_\eta, \qquad \sigma_{\eta,0} = \sigma_\eta.$$

Then we have the following robustness theorem.

Theorem 3.7 (Theorem 2, [BCH23]). *Let $\delta > 0$ and $\eta > 0$ be such that Theorem 3.2 holds, i.e.,*

$$\sigma_{\eta,0} \leq -\ln(2) + \delta.$$

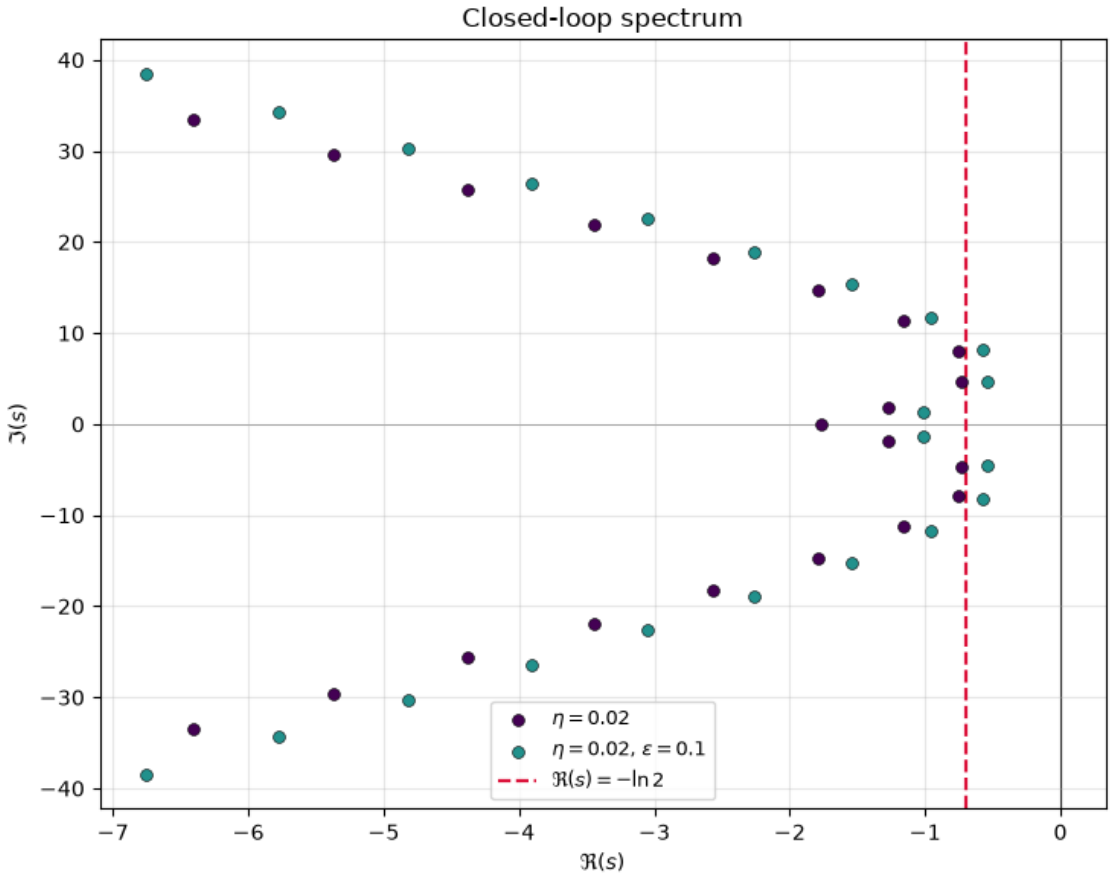


Figure 3.5: Closed-loop spectrum of (3.12) with $\eta = 0.02$ and $\varepsilon = 0.1$

Then there exists $\varepsilon_1 > 0$ *such that, for every* $\varepsilon \in (-\varepsilon_1, \varepsilon_1)$*, the perturbed maximal spectral abscissa satisfies*

$$\sigma_{\eta,\varepsilon} \leq -\ln(2) + 2\delta.$$

Proof. Assume by contradiction that the theorem is false. That is, for every $\varepsilon_1 > 0$ there exists $\varepsilon \in (-\varepsilon_1, \varepsilon_1)$ such that

$$\sigma_{\eta,\varepsilon} > -\ln(2) + 2\delta,$$

and hence there exists $s \in \mathcal{S}_{\eta,\varepsilon}$ such that

$$\Re(s) > -\ln(2) + 2\delta.$$

Then we can define a sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ with $\lim_{n\to\infty} \varepsilon_n = 0$, which has an associated sequence $(s_n)_{n\in\mathbb{N}}$ such that

$$\Re(s_n) > -\ln(2) + 2\delta, \quad s_n \in \mathcal{S}_{\eta,\varepsilon_n}, \ \forall n \in \mathbb{N}. \tag{3.30}$$

We split the proof into two cases: when $(s_n)_{n\in\mathbb{N}}$ is bounded and when it is not.

In the first case, we can extract a subsequence (still denoted $(s_n)_{n\in\mathbb{N}}$) converging to $\overline{s} \in \mathbb{C}$. Since $\varepsilon_n \to 0$, we can pass to the limit in the characteristic equation (3.29) and deduce that $\overline{s} \in \mathcal{S}_{\eta,0} = \mathcal{S}_\eta$, and by hypothesis on δ and η,

$$\Re(\overline{s}) \leq -\ln(2) + \delta. \tag{3.31}$$

On the other hand, passing to the limit in (3.30),

$$\Re(\overline{s}) \geq -\ln(2) + 2\delta > -\ln(2) + \delta,$$

which contradicts (3.31).

We now consider the second case. Suppose $(s_n)_{n\in\mathbb{N}}$ is unbounded. Then we can extract a subsequence, denoted the same way, such that $|s_n| \to \infty$. Write $s_n = r_n e^{i\theta_n}$ with $r_n \in \mathbb{R}_+$ and $\theta_n \in [-\pi, \pi)$. Since $r_n = |s_n| \to +\infty$ and by (3.30), for n large enough $\theta_n \in (-5\pi/8, 5\pi/8)$.

Write $(1+\varepsilon_n)^2 + 4\eta s_n = r_n^1 e^{i\theta_n^1}$, again with $r_n^1 \in \mathbb{R}_+$ and $\theta_n^1 \in [-\pi, \pi)$. For n large enough, depending on η, $r_n^1 \to +\infty$ and $\theta_n^1 \in (-3\pi/4, 3\pi/4)$. Hence

$$\sqrt{(1+\varepsilon_n)^2 + 4\eta s_n} = \sqrt{r_n^1} e^{i\theta_n^1/2},$$

with $\theta_n^1/2 \in (-3\pi/8, 3\pi/8)$, which implies $\Re(\sqrt{(1+\varepsilon_n)^2 + 4\eta s_n}) \to +\infty$. Using the definition of $\lambda_{1,2}$,

$$\Re(\lambda_1(s_n)) \to +\infty, \qquad \Re(\lambda_2(s_n)) \to -\infty,$$

and, after a short computation,

$$f_{\eta,\varepsilon_n}(s_n) = 2\left[\left(\frac{(1+\varepsilon_n)}{\sqrt{(1+\varepsilon_n)^2 + 4\eta s_n}} + 1\right) e^{-\lambda_1(s_n)} + \left(\frac{(1+\varepsilon_n)}{\sqrt{(1+\varepsilon_n)^2 + 4\eta s_n}} - 1\right) e^{-\lambda_2(s_n)}\right]^{-1}.$$

Taking the limit $n \to \infty$, we get $f_{\eta,\varepsilon_n}(s_n) \to 0$, since $\left(\frac{(1+\varepsilon_n)}{\sqrt{(1+\varepsilon_n)^2+4\eta s_n}}\right) \to 0$. Moreover, thanks to (3.30), e^{-s_n} remains under control, and we may take the limit in the characteristic equation,

$$0 = f_{\eta,\varepsilon_n}^2(s_n) - e^{-s_n} f_{\eta,\varepsilon_n}(s_n) - 1 \to -1,$$

which is a contradiction.

That is, in both cases we obtain a contradiction, which means there exists $\varepsilon_1 > 0$ such that $\sigma_{\eta,\varepsilon} \leq -\ln(2) + 2\delta$ for every $\varepsilon \in (-\varepsilon_1, \varepsilon_1)$. □

3.6 Comparison with a simpler system

In this section we study a simpler system which is very close to the previous one and motivates Chapter 4. We start with a system without diffusion nor uncertainty,

$$\begin{cases} \partial_t y(t,x) + \partial_x y(t,x) = 0, \\ \partial_t \tilde{y}(t,x) + \partial_x \tilde{y}(t,x) = 0, \\ \begin{pmatrix} y(t,0) \\ \tilde{y}(t,0) \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} y(t,1) \\ \tilde{y}(t,1) \end{pmatrix}. \end{cases} \tag{3.32}$$

Following the calculations from section 3.3, we obtain the characteristic equation $e^{2s} = 0$, meaning that the system is stable for any decay rate. In fact, it is finite time stable since for any time $t \geq 1$, $\tilde{y}(t,\cdot) = y(t,\cdot)$ on $[0,1]$. Thus, from the condition $y(t,0) = 0$ at $x = 0$, we get that $y(t,\cdot) \equiv 0$ for $t \geq 2$. The problem is that

$$\overline{\rho}(\mathbf{K}) = \sqrt{2}, \qquad \mathbf{K} = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix},$$

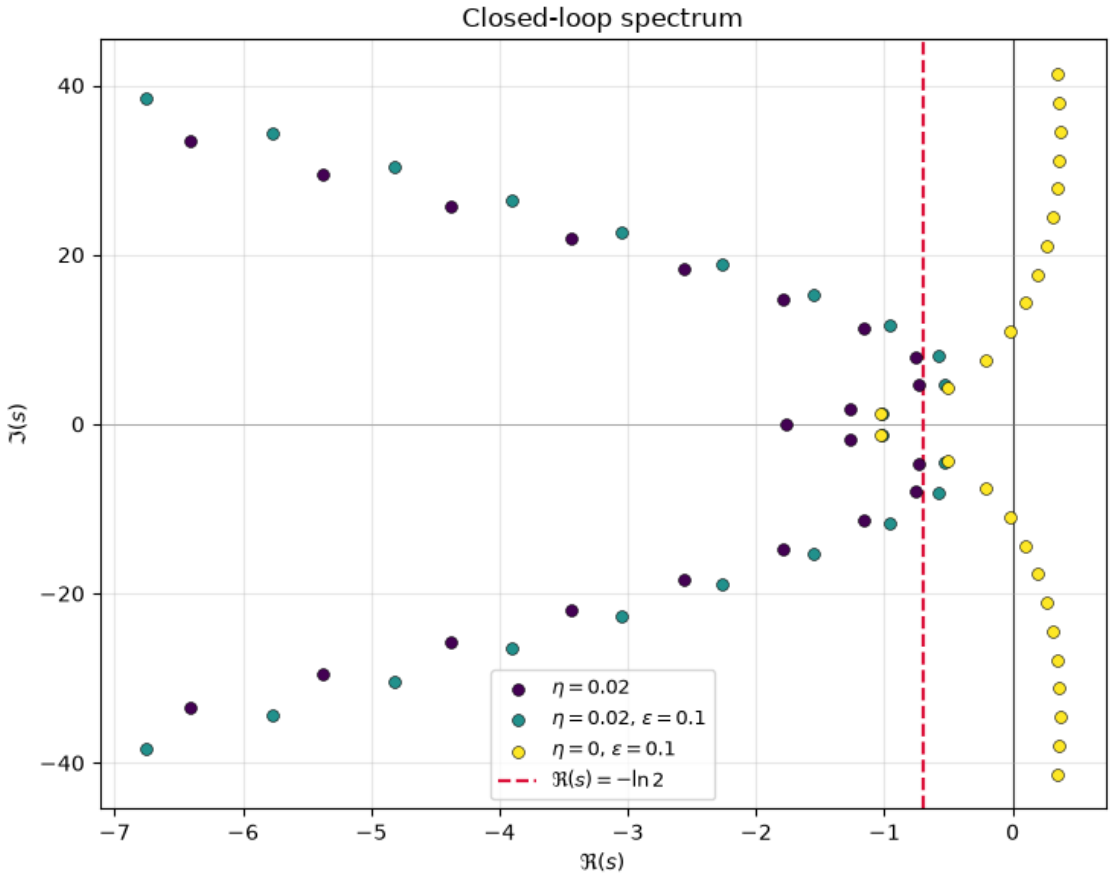


Figure 3.6: Closed-loop spectrum of (3.28)

so the stability is not robust with respect to delay inaccuracy.

By the discussion made on the previous sections, it is natural to ask if the addition of a diffusion term in (3.32) in just the first equation allows us to strengthen the system stability. The new system becomes

$$\begin{cases} \partial_t y(t,x) + \partial_x y(t,x) - \eta \partial_{xx}^2 y(t,x) = 0, \\ \partial_t \tilde{y}(t,x) + \partial_x \tilde{y}(t,x) = 0, \\ y(t,0) = y(t,1) - \tilde{y}(t,1), \\ \tilde{y}(t,0) = y(t,0), \\ \partial_x y(t,1) = 0. \end{cases} \tag{3.33}$$

Here we are still using $\eta > 0$ as a diffusion parameter. Following similar calculations as in Section 3.3, we find the transfer functions for the two equations. First, note that the solution to the transport equation takes the form

$$\tilde{y}(s,x) = \tilde{y}(s,0)e^{-sx}, \qquad x \in (0,1).$$

Since $\tilde{y}(s,0) = y(s,0)$,

$$\tilde{y}(s,1) = e^{-s}y(s,0).$$

On the other hand, for the modified transport equation we have

$$y(s,x) = A(s)e^{\lambda_1(s)x} + B(s)e^{\lambda_2(s)x}, \qquad \forall x \in (0,1).$$

Using the boundary condition $\partial_x y(s,1) = 0$, we get

$$B(s) = -\frac{\lambda_1(s)A(s)e^{\lambda_1(s)}}{\lambda_2(s)e^{\lambda_2(s)}}.$$

This yields the transfer function $f_\eta(s)$

$$f_\eta(s) = \frac{y(s,1)}{y(s,0)} = \frac{e^{\lambda_1(s)} - \frac{\lambda_1(s)e^{\lambda_1(s)}}{\lambda_2(s)e^{\lambda_2(s)}}}{\lambda_1(s) - \frac{\lambda_2(s)}{\lambda_2(s)e^{\lambda_2(s)}}} = \frac{(\lambda_1(s) - \lambda_2(s))e^{\lambda_1(s)+\lambda_2(s)}}{\lambda_1(s)e^{\lambda_1(s)} - \lambda_2(s)e^{\lambda_2(s)}}.$$

By Figure 3.2, we get that since $y(s,0) = y(s,1) - \tilde{y}(s,1) = y(s,1) - e^{-s}y(s,0)$,

$$(1 + e^{-s})y(s,0) = y(s,1).$$

In other words, since we are looking for a non-trivial mode, we need $A(s) \not\equiv 0$, which means that $f_\eta(s) = 1 + e^{-s}$. Equivalently,

$$F_\eta(s) := \frac{1 + e^{-s}}{f_\eta(s)} = \frac{(\lambda_1(s)e^{\lambda_1(s)} - \lambda_2(s)e^{\lambda_2(s)})(1 + e^{-s})}{(\lambda_1(s) - \lambda_2(s))e^{\lambda_1(s)+\lambda_2(s)}} = 1. \tag{3.34}$$

$$F_\eta(s) = \frac{f_\eta(s)}{1 - f_\eta(s)g_0(s)}$$

$$F_\eta(s) - 1 = 0, \qquad F_\eta(s) = \frac{(\lambda_1(s)e^{\lambda_1(s)} - \lambda_2(s)e^{\lambda_2(s)})(1 + e^{-s})}{(\lambda_1(s) - \lambda_2(s))e^{\lambda_1(s)+\lambda_2(s)}},$$

where $\lambda_{1,2}(s)$ are given by (3.9). For a given value of η, we use the same notation for this problem,

$$\mathcal{S}_\eta = \{s \in \mathbb{C} : F_\eta(s) - 1 = 0\},$$
$$\sigma_\eta = \sup\{\Re(s) : s \in \mathcal{S}_\eta\}.$$

We then have the surprising observation that a slight diffusion in the system has, in this case, a destabilizing effect as one could see in the Figure 3.6. This leads to the following conjecture.

Conjecture 2. For all $\varepsilon > 0$, there exists $\eta_1 > 0$ such that for all $\eta \in (0, \eta_1)$ the maximal spectral abscissa satisfies the inequality $\sigma_\eta > -\varepsilon$.

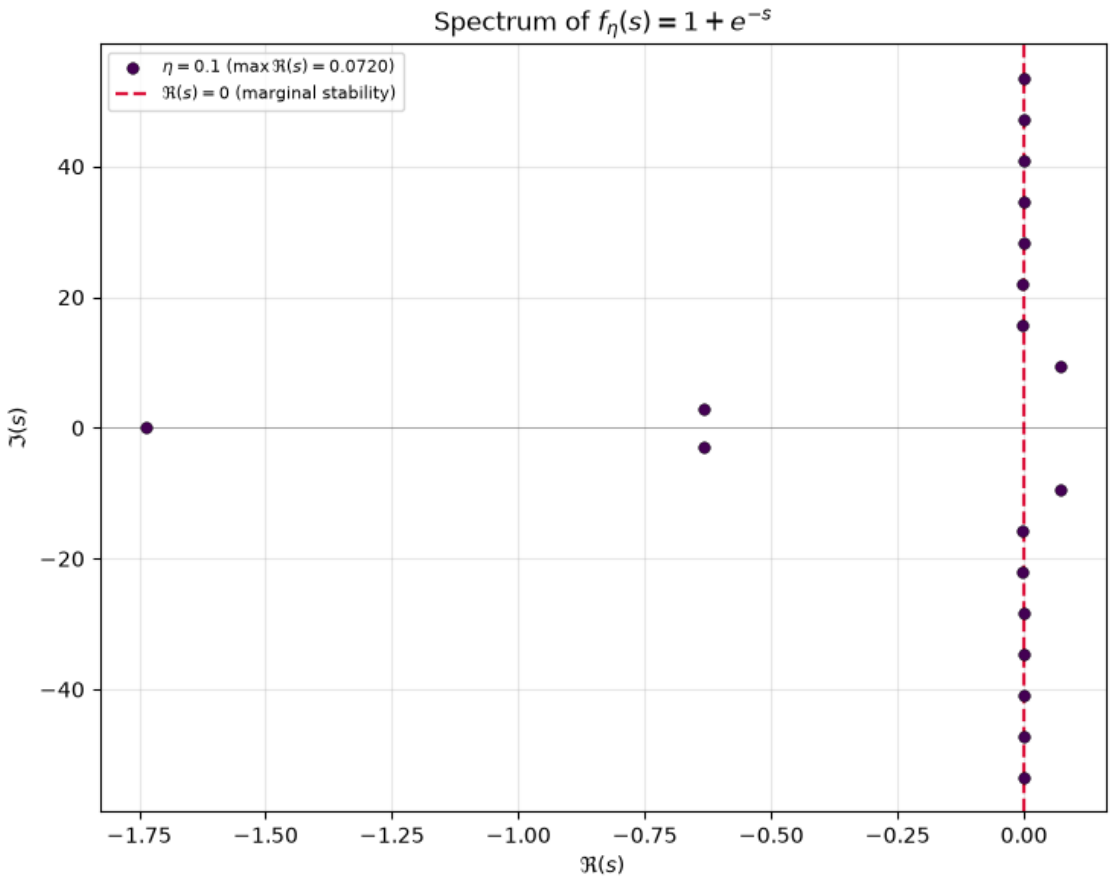


Figure 3.7: Closed-loop spectrum of (3.34)

Chapter 4

Stability and robustness for a hyperbolic system with internal coupling

This Chapter is based on [BCH25b].

4.1 Motivation

The stabilization of hyperbolic systems is not robust enough to handle problems where there is a delay between the input provided by the control and the output of the measurement. This occurs when stabilization requires a dynamic feedback law for the control that includes delayed values of the output, which may also depend on the state of an observer. In this case, the lack of robustness means that small changes in the model can result in unstable solutions of a closed-loop system that, a priori, is technically stabilizable [BCH25b].

Several recent works have approached the robustness of boundary-feedback control for coupled hyperbolic systems through the transfer function of the closed-loop system, generally in the collocated setting. Auriol, Aarsnes, Martin and Di Meglio [Aur+18a] design a delay-robust backstepping controller for two heterodirectional collocated hyperbolic PDEs and derive an explicit robustness condition directly from the transfer function of the closed loop. Auriol and Bresch-Pietri [AB22] extend this transfer function/backstepping approach to an underactuated network of interconnected $n+m$ collocated hyperbolic PDE systems, while Auriol, Bribiesca Argomedo, Bou Saba, Di Loreto and Di Meglio [Aur+18b] treat a collocated hyperbolic PDE-ODE system with constant transport velocities, in which the delay-robust feedback is produced by an ODE. The anti-located configuration that is the subject of the present thesis is comparatively less explored with these tools. Guo and Xu [GX07] stabilize the wave equation by an anti-located boundary feedback using transfer function and approximation methods, and Hasan and Tang [HT22] stabilize a coupled Burgers PDE-ODE system, again with an anti-located ODE feedback, but through a backstepping and Lyapunov argument rather than a transfer function analysis. More recently, Karafyllis and Krstic [KK22] study a related problem of viscosity-based boundary stabilization for a viscous liquid transfer system, closer in spirit to the role played by viscosity in this chapter, although again without a transfer function analysis. The approach of this chapter follows the transfer function tradition of [Aur+18a; AB22; Aur+18b; GX07], adapted to the anti-located, internally coupled density-flow system introduced below, and the lower bound obtained in Section 4.5 relies on topological degree theory, for which [FG95, pages 45 and 46] remains a standard reference for applications like this.

4.2 Presentation and Motivation of the Control Problem

Consider a hyperbolic system of two linear conservation laws in one spatial dimension, with the form

$$\begin{aligned} H_t + (HV)_x &= 0, \\ V_t + \left(gH + \frac{V^2}{2}\right)_x &= 0, \end{aligned}$$

where $t \geq 0$ is the time coordinate, $x \in [0, L]$ is the spatial coordinate, and $H(t, x)$, $V(t, x)$ denote the water level and flow velocity of some quantity of interest. This is the classical Saint-Venant system, for which Hayat and Shang [HS19] construct a quadratic Lyapunov function valid under arbitrary friction and a space-varying slope, complementing the transfer-function viewpoint adopted here. Linearizing around a constant equilibrium and writing the resulting 2×2 system in terms of the density $H(t, x)$ and flow rate $Q(t, x)$, we obtain the density-flow system,

$$\begin{aligned} \partial_t H + \partial_x Q &= 0, \\ \partial_t Q + c_1 c_2 \partial_x H - (c_1 - c_2)\partial_x Q &= 0, \end{aligned}$$

where c_1 and c_2 are positive constants. In these equations $H(t, x)$ represents the density and $Q(t, x)$ the flux density of the quantity of interest, so the system is known in the literature as a density-flow system.

In this chapter we focus on solutions of the Cauchy problem with initial conditions

$$H(0, x), \qquad Q(0, x), \qquad x \in [0, L],$$

and two boundary conditions

$$Q(t, 0) = Q_0(t), \qquad Q(t, L) = Q^*, \qquad t \geq 0,$$

where $Q_0(t)$ is a time-varying flow input that can be assigned by the operator, while Q^* is a constant outgoing flow, imposed arbitrarily by the operating conditions.

Since for every pair of constant states

$$H(t, x) = H^*, \qquad Q(t, x) = Q^*,$$

for all $(t, x) \in [0, \infty) \times [0, L]$, the state of the system does not change. Hence the Cauchy problem under study has a continuum of non-isolated equilibrium points, which are not asymptotically stable. This is why there is interest in studying the feedback-type stabilization of the system.

For our purposes, we focus on the case where $Q^* = 0$ and the two characteristic velocities are equal, i.e., $c_1 = c_2 = c > 0$. In that case the model is

$$\begin{cases} \partial_t H + \partial_x Q = 0, \\ \partial_t Q + c^2 \partial_x H = 0, \end{cases}$$

where c is the wave celerity. We introduce the notation

$$h(t, x) = H(t, x) - H^*, \qquad q(t, x) = Q(t, x) - Q^*.$$

Hence the open-loop system can be written as

$$\begin{cases} \partial_t h(t,x) - \partial_x q(t,x) = 0, \\ \partial_t q(t,x) + c^2 \partial_x h(t,x) = 0, \\ q(t,0) = U(t), \\ q(t,L) = 0, \\ Y(t) = h(t,L), \end{cases}$$

where $U(t)$ is the control input and $Y(t)$ is the measured output. In the frequency domain, i.e. after applying the Laplace transform in time, we have the system

$$\begin{cases} sh(s,x) + \partial_x q(s,x) = 0, \\ sq(s,x) + c^2 \partial_x h(s,x) = 0, \\ q(s,0) = U(s), \\ q(s,L) = 0, \\ Y(s) = h(s,L). \end{cases} \tag{4.1}$$

Here $s \in \mathbb{C}$ is the Laplace variable. Differentiating the first equation with respect to x, we obtain

$$s\partial_x h(s,x) = -\partial_{xx}^2 q(s,x),$$

and using this relation in the second equation gives

$$s^2 q(s,x) - c^2 \partial_{xx}^2 q(s,x) = 0.$$

Then, for $s \neq 0$, the solution of this ODE has the form

$$q(s,x) = A(s)e^{sx/c} + B(s)e^{-sx/c}.$$

Using the boundary conditions, we obtain

$$q(s,0) = A(s) + B(s) = U(s),$$

and

$$q(s,L) = A(s)e^{s\tau} + B(s)e^{-s\tau} = 0, \qquad \tau = \frac{L}{c}.$$

By the observation, we have

$$Y(s) = h(s,L) = -\frac{1}{s}\partial_x q(s,L) = \frac{1}{c}(-A(s)e^{s\tau} + B(s)e^{-s\tau}),$$

so eliminating $A(s)$ and $B(s)$ from the equations, the transfer function for the open-loop control is

$$G_0(s) = \frac{Y(s)}{U(s)} = \frac{2e^{-s\tau}}{c(1-e^{-2s\tau})}, \tag{4.2}$$

whose poles are the solutions of the equation

$$e^{2s\tau} - 1 = 0.$$

These poles have imaginary part greater than or equal to 0, so the open-loop system is not stable. Moreover, the system is not bounded-input bounded-output stable. If we choose $U(t) = 0$ to have a constant state, the system is also not asymptotically stable.

We claim that this system cannot be stabilized with a simple proportional output feedback of the form

$$U(t) = -k_c Y(t), \qquad k_c \neq 0.$$

Indeed, system (4.1) with the control above has the following characteristic equation,

$$e^{2s\tau} + 2(k_c/c)e^{s\tau} - 1 = 0.$$

Solving the quadratic,

$$e^{s\tau} = -(k_c/c) \pm \sqrt{1 + (k_c/c)^2}.$$

Hence, for $k_c \neq 0$, there are infinitely many poles $\sigma + i\omega$ located on two vertical lines with real part

$$\sigma = c\ln(\sqrt{1 + (k_c/c)^2} + |k_c/c|) > 0, \qquad \sigma = c\ln(\sqrt{1 + (k_c/c)^2} - |k_c/c|) < 0.$$

It follows that the system cannot be stabilized with this static control, since half of the poles of the closed-loop system have strictly positive real part. This motivates the idea of proving stabilization using a dynamic control.

Thanks to the static feedback, transfer function (4.2) can be rewritten as a delay-difference equation,

$$Y(t) - Y(t - 2\tau) = \frac{2}{c}U(t - \tau).$$

Hence the system can be stabilized with a delayed output feedback of the form

$$U(t) = -\frac{c}{2}Y(t - \tau), \tag{4.3}$$

so that the closed-loop system reduces to $Y(t) = 0$ for all t after the initial instant. We know that this class of hyperbolic problems is sensitive to small errors when delayed controls are used. This lack of robustness can be shown by rewriting the system in Riemann coordinates,

$$\begin{aligned} y_1(t,x) &:= \frac{1}{2}(q(t,x) + ch(t,x)), \\ y_2(t,x) &:= \frac{1}{2}(q(t,x) - ch(t,x)). \end{aligned}$$

In these coordinates, system (4.1) is written as

$$\begin{cases} \partial_t y_1(t,x) + c\partial_x y_1(t,x) = 0, \\ \partial_t y_2(t,x) - c\partial_x y_2(t,x) = 0, \\ y_1(t,0) = -y_2(t,0) + U(t), \\ y_2(t,L) = -y_1(t,L), \\ Y(t) = \frac{2}{c}y_1(t,L). \end{cases} \tag{4.4}$$

Moreover, a representation of the dynamic control (4.3) is given by

$$\begin{cases} \partial_t y_3(t,x) + c\partial_x y_3(t,x) = 0, \\ y_3(t,0) = Y(t), \\ U(t) = -\frac{c}{2}y_3(t,L). \end{cases} \tag{4.5}$$

Closing system (4.4) with the differential representation of the dynamic control (4.5), we obtain the system

$$\begin{pmatrix} y_1(t,0) \\ y_2(t,L) \\ y_3(t,0) \end{pmatrix} = \begin{pmatrix} 0 & -1 & -c/2 \\ -1 & 0 & 0 \\ 2/c & 0 & 0 \end{pmatrix} \begin{pmatrix} y_1(t,L) \\ y_2(t,0) \\ y_3(t,L) \end{pmatrix}. \tag{4.6}$$

Denoting the matrix above by $\mathbf{K}$, its spectral radius is given by

$$\overline{\rho}(\mathbf{K}) = \sqrt{2}, \quad c > 0,$$

where

$$\overline{\rho}(\mathbf{K}) = \max\{\rho(\mathrm{diag}(e^{-i\theta_1}, e^{-i\theta_2}, e^{-i\theta_3})\mathbf{K}),\ (\theta_1, \theta_2, \theta_3)^\top \in \mathbb{R}^3\}.$$

We know that $\overline{\rho}(\mathbf{K}) < 1$ is a necessary and sufficient condition for strong stability, i.e., for robustness against certain perturbations. This shows the necessity of the addition of some term that helps the system to be stable under this change on the nominal velocities. The contribution of the paper is to show that this lack of robustness is actually an artefact which stems from the assumption that the viscosity can be neglected when modelling the fluid system (4.1). It is shown that the robustness of the output feedback stabilization is recovered as soon as an arbitrary small diffusion is present in the system even with the simple delay control law.

4.3 The Open-Loop Control System with Viscosity

We modify system (4.1) with a small viscosity. For simplicity and without loss of generality, we take $L = 1$ and $\tau = L/c = 1$. Hence the dynamics of the system in the time domain is given by

$$\begin{cases} \partial_t h(t,x) + \partial_x q(t,x) = 0, \\ \partial_t q(t,x) + \partial_x h(t,x) - \eta \partial_{xx}^2 q(t,x) = 0, \\ q(t,0) = U(t), \\ q(t,1) = 0, \\ Y(t) = h(t,1), \end{cases} \tag{4.7}$$

where $\eta > 0$ and $\eta\partial_{xx}^2 q$ is the diffusion term. It follows that any pair of constant states $h(t,x) = H^*$, $q(t,x) = 0$ is an equilibrium corresponding to $U(t) = 0$. Hence the system has a continuum of non-isolated equilibrium points and is not asymptotically stable.

Remark 4.1. It should be noted that this system differs from (3.7) in two ways. The first is that the coupling is different, since now it is being effective at the equations level and not on the boundary conditions. The other difference is that we are adding just one diffusion term, so in this sense this is a stronger result, because the same upper (and now, also the lower) bound will be achieved. See Section 3.6, where the coupling on the boundary conditions yields a destabilizing effect on the system.

In the frequency domain we can write

$$\begin{cases} sh(s,x) + \partial_x q(s,x) = 0, \\ sq(s,x) + \partial_x h(s,x) - \eta \partial_{xx}^2 q(s,x) = 0, \\ q(s,0) = U(s), \\ q(s,1) = 0, \\ Y(s) = h(s,1). \end{cases} \tag{4.8}$$

Differentiating the first equation with respect to x and substituting into the second, we obtain

$$s^2 q(s,x) - (1+\eta s)\partial_{xx}^2 q(s,x) = 0, \tag{4.9}$$

so that for every $s \neq 0$, the solution of the ODE is

$$q(s,x) = A(s)e^{\lambda(s)x} + B(s)e^{-\lambda(s)x},$$

where $\pm\lambda(s)$ solve

$$(1+\eta s)\lambda^2 - s^2 = 0. \tag{4.10}$$

Noting that

$$Y(s) = h(s,1) = -\frac{1}{s}\partial_x q(s,1) = -\frac{\lambda(s)}{s}(A(s)e^{\lambda(s)} - B(s)e^{-\lambda(s)}),$$

we can solve the system

$$\begin{cases} q(s,0) = A(s) + B(s) = U(s), \\ q(s,1) = A(s)e^{\lambda(s)} + B(s)e^{-\lambda(s)} = 0, \\ Y(s) = -\frac{\lambda(s)}{s}(A(s)e^{\lambda(s)} - B(s)e^{-\lambda(s)}). \end{cases} \tag{4.11}$$

Eliminating $A(s), B(s)$, we obtain the following expression for the transfer function,

$$G(s) = \frac{Y(s)}{U(s)} = \frac{2e^{\lambda(s)}}{\sqrt{1+\eta s}\,(e^{2\lambda(s)} - 1)}. \tag{4.12}$$

Using this, we note that for every $\eta \neq 0$, the transfer function $G(s)$ has a pole at 0. In other words, system (4.7) is unstable because it is unbounded. Some remarks are in order.

Remark 4.2. This system is not in plant form. This means that we cannot try to get the system transfer equation in terms of the channel transfer equations as in Figure 3.1. This is mainly due to the coupling being in the equations rather than in the boundary conditions.

The other difficulty with respect to the original problem is that the expression for the system transfer function seems quite different and hard to manipulate. In fact, there is no change of variable that yields a second order algebraic equation to be studied.

4.4 Stability of the Closed-Loop System with Viscosity

If we consider the control $U(t) = -\frac{1}{2}Y(t-1)$, system (4.4) becomes closed loop. In the frequency domain, this feedback is given by

$$U(s) = -\frac{1}{2}e^{-s}Y(s).$$

With this, we can solve system (4.11) under the above feedback, eliminating $A(s)$ and $B(s)$ again, to obtain the system

$$\begin{pmatrix} 2s - \lambda e^{-s}e^{\lambda} & 2s + \lambda e^{-s}e^{-\lambda} \\ e^{\lambda} & e^{-\lambda} \end{pmatrix} \begin{pmatrix} A(s) \\ B(s) \end{pmatrix} = 0.$$

Denoting the matrix above by $M(s)$, the characteristic equation $\det M(s) = 0$ of the closed-loop system is

$$\mathcal{G}(\eta, s) = e^s\sqrt{1+\eta s}\left(e^{s/\sqrt{1+\eta s}} - e^{-s/\sqrt{1+\eta s}}\right) + 1 = 0.$$

Our goal is to prove spectral stability of this closed-loop system. Given a viscosity value $\eta > 0$, we denote by $\mathcal{S}_\eta$ the spectrum of the closed-loop system, the set of poles that are roots of the characteristic equation,

$$\mathcal{S}_\eta = \{s \in \mathbb{C} : \ \mathcal{G}(\eta, s) = 0\}.$$

Moreover, the maximal spectral abscissa is the supremum of the real parts of the spectrum,

$$\sigma_\eta := \sup\{\Re(s) : \ s \in \mathcal{S}_\eta\}.$$

We know that a necessary condition to ensure asymptotic stability of the system is that the maximal spectral abscissa be negative. This could be checked by numerical implementation, and it could be seen that the maximal spectral abscissa σ_η appears to be approximately $-\ln(2)$ and independent of η, in the same fashion as in Chapter 3. This intuitive observation is in accordance with the following theorem.

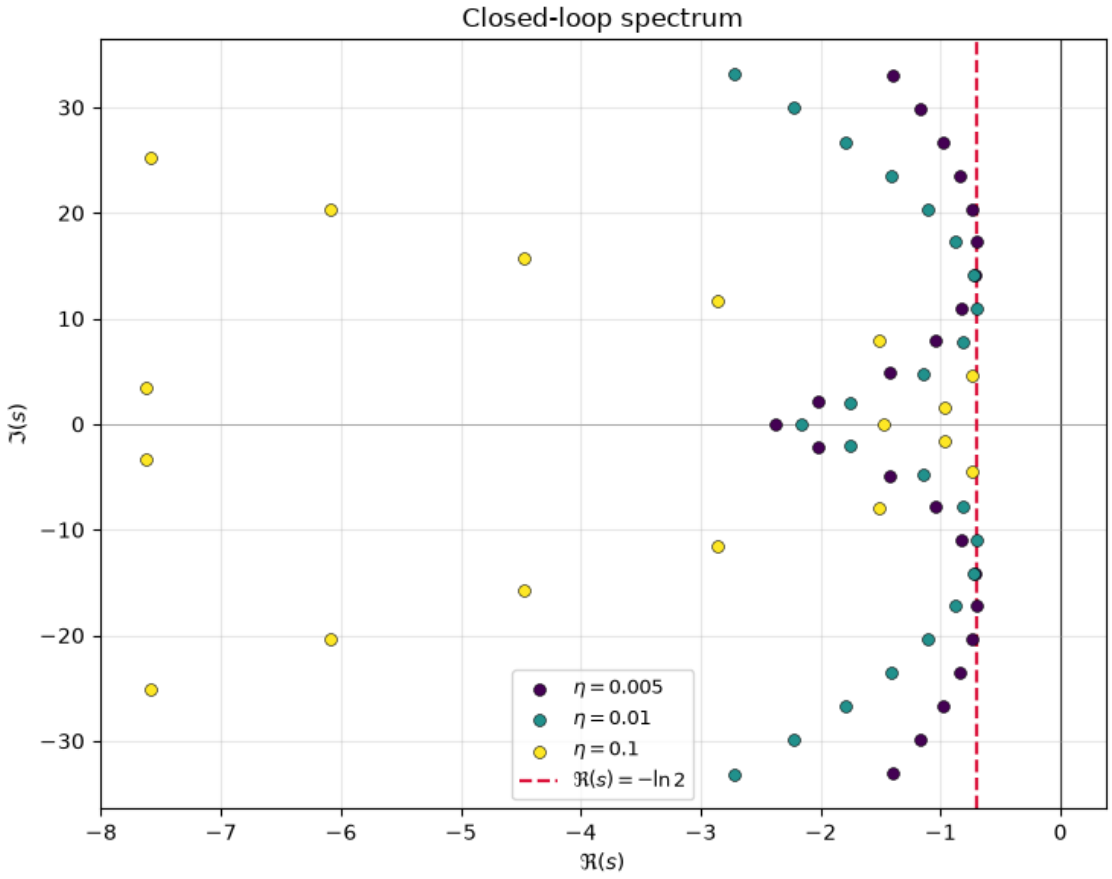


Figure 4.1: Closed-loop spectrum of (4.7)

Theorem 4.3 (Theorem 1, [BCH25b]). *For every $\delta \in (0, \ln(2))$, there exists $\eta^* > 0$ such that*

$$-\ln(2) - \delta < \sigma_\eta < -\ln(2) + \delta, \qquad \forall \eta \in (0, \eta^*]$$

Remark 4.4. Obtaining a negative upper bound for the real part of the spectrum is enough to prove decay if the system satisfies the spectral mapping property. This is usually true for one-dimensional physical systems, but it has not been proved for this system. It is expected that this can be proved using the Gearhart–Prüss theorem [CZ95].

Remark 4.5 (Important remark)**.** It should be noted that this theorem states also a lower bound for the maximal spectral abscissa, which is not proved on Theorem 3.2 on Chapter 3. This solves, in some way, Conjecture 1 from the past chapter for this new system. As we will see at the end of the chapter, the method developed here helps to solve Conjecture 1 in the same setup of Chapter 3.

To prove Theorem 4.3 we set up some notation. Consider a sequence

$$(\eta_n)_{n\in\mathbb{N}} : \eta_n > 0,\ \forall n \in \mathbb{N}, \quad \lim_{n\to+\infty} \eta_n = 0^+,$$

and an associated sequence of poles

$$(s_n)_{n\in\mathbb{N}} : \quad s_n \in \mathcal{S}_{\eta_n},\ \forall n \in \mathbb{N}.$$

From the characteristic equation,

$$\mathcal{G}(\eta_n, s_n) = e^{s_n}\theta_n \left(e^{s_n/\theta_n} - e^{-s_n/\theta_n}\right) + 1 = 0, \quad \forall n \in \mathbb{N}, \tag{4.13}$$

where we denote

$$\theta_n = \sqrt{1 + \eta_n s_n}.$$

To prove Theorem 4.3 we use accumulation points of $(s_n)_{n\in\mathbb{N}}$ as $n \to +\infty$, i.e., as $\eta_n \to 0^+$. By definition, $\overline{s}$ is an accumulation point of $(s_n)_{n\in\mathbb{N}}$ if and only if there is a subsequence converging to $\overline{s}$. With a slight abuse of notation, we simply write $s_n \to \overline{s}$.

The proof of Theorem 4.3 follows from the next two structural lemmas.

Lemma 4.6 (Lemma 1, [BCH25b])**.** *Let $(\eta_n)_{n\in\mathbb{N}}$ be a sequence of viscosities and $(s_n)_{n\in\mathbb{N}}$ the associated sequence of poles. Denote $\sigma_n = \Re(s_n)$ and $\omega_n = \Im(s_n)$. Consider the induced sequences $(\sigma_n)_{n\in\mathbb{N}}$ and $(\eta_n\omega_n^2)_{n\in\mathbb{N}}$. Then*

1. *If $\overline{\sigma}$ is an accumulation point of $(\sigma_n)_{n\in\mathbb{N}}$, then $\overline{\sigma} \in [-\infty, +\infty)$.*
2. *If $\overline{\sigma} \in (-\infty, +\infty)$ and 2θ is an accumulation point of $(\eta_n\omega_n^2)_{n\in\mathbb{N}}$, then $2\theta \in [0, +\infty)$.*

Proof. To prove the first part, assume by contradiction that $\overline{\sigma} = +\infty$ and therefore there exists $N \in \mathbb{N}$ such that $\sigma_n > 0$ for every $n \geq N$. This implies that

$$|e^{s_n}| \to +\infty, \qquad \left|\sqrt{1 + \eta_n s_n}\right| \geq 1. \tag{4.14}$$

Then it follows from (4.13) that

$$\lim_{n\to+\infty} \left(e^{s_n/\sqrt{1+\eta_n s_n}} - e^{-s_n/\sqrt{1+\eta_n s_n}}\right) = 0. \tag{4.15}$$

In the following we will write $X_n = e^{s_n/\sqrt{1+\eta_n s_n}}$. Since the limit (4.15) is 0, we get that X_n and X_n^{-1} are bounded. Therefore since

$$||X_n| - |X_n|^{-1}| \leq |X_n - X_n^{-1}|,$$

(4.15) implies that 1 is the only possible adherence value for $|X_n|$, and therefore

$$|e^{s_n/\sqrt{1+\eta_n s_n}}| \to 1. \tag{4.16}$$

Now we denote

$$z_n = \frac{1}{\sqrt{1+\eta_n s_n}} = \frac{\sqrt{1+\eta_n\sigma_n - i\eta_n\omega_n}}{\sqrt{(1+\eta_n\sigma_n)^2+(\eta_n\omega_n)^2}}, \tag{4.17}$$

where $s_n = \sigma_n + i\omega_n$, which implies that since $1+\eta_n\sigma_n$,

$$0 \le |\Im(z_n)| \le \frac{1}{\sqrt{2}} \frac{1}{((1+\eta_n\sigma_n)^2+(\eta_n\omega_n)^2)^{1/4}} = \frac{1}{\sqrt{2}}|z_n|,$$

and therefore,

$$0 \le |\Im(z_n)| \le \frac{1}{\sqrt{2}}|z_n| \le \Re(z_n) \le |z_n|. \tag{4.18}$$

With these notations, we rewrite (4.16) as

$$\left|e^{\sigma_n\Re(z_n)-\omega_n\Im(z_n)+i(\sigma_n\Im(z_n)+\omega_n\Re(z_n))}\right| \to 1, \tag{4.19}$$

thus,

$$\sigma_n\Re(z_n) - \omega_n\Im(z_n) \to 0. \tag{4.20}$$

We work on three cases for ω_n. First suppose that $\omega_n \to 0$, then from (4.18), $\omega_n\Im(z_n) \to 0$ and $\sigma\Re(z_n) \to +\infty$, which contradicts (4.20).

Now, by symmetry, suppose that $\omega_n > 0$. The case $\omega_n < 0$ is analogous. From (4.17), we see that $\Re(z_n) > 0$ and $\Im(z_n) < 0$. Since $\sigma_n > 0$, we deduce from (4.20) that

$$|\sigma_n\Re(z_n)| + |\omega_n\Im(z_n)| \to 0. \tag{4.21}$$

Using (4.21) together with (4.18) gives

$$|\sigma_n||z_n| + |\omega_n\Im(z_n)| \to 0. \tag{4.22}$$

If there exists $C > 0$ such that for n sufficiently large $|\eta_n\omega_n| \le C(1+|\eta_n\sigma_n|)$, then

$$|\sigma_n||z_n| = \frac{\sigma_n}{((1+\eta_n\sigma_n)^2+(\eta_n\omega_n)^2)^{1/4}} \tag{4.23}$$

$$\le \frac{\sigma_n}{(1+C^2)^{1/4}(1+\eta_n\sigma_n)^{1/2}} \to +\infty, \tag{4.24}$$

and this contradicts (4.22). Thus, for n sufficiently large we can assume $|\eta_n\omega_n| > 1+|\eta_n\sigma_n|$ up to a subsequence and therefore, by (4.17) there exists $c > 0$ independent of n such that $|\Im(z_n)| \ge c|z_n|$. This, combined with (4.22) gives

$$(|\sigma_n| + |\omega_n|)|z_n| \to 0. \tag{4.25}$$

From the definition of z_n we have $s_n/\sqrt{1+\eta_n s_n} \to 0$. Taking the square, this leads to

$$\frac{|s_n|}{\left|\frac{1}{s_n}+\eta_n\right|} \to 0.$$

Since $|s_n| \geq \sigma_n \to +\infty$ and η_n is bounded, we obtain a contradiction, which concludes the proof of the first part.

To prove the second point, we know that $2\theta \geq 0$ since $(\eta_n \omega_n^2)_{n\in\mathbb{N}}$ is positive for all n. Assume by contradiction that $2\theta = +\infty$. We have

$$\frac{s_n}{\sqrt{1+\eta_n s_n}} = \frac{(\sigma_n + i\omega_n)\sqrt{1+\sigma_n\eta_n - \omega_n\eta_n}}{\sqrt{(1+\eta_n\sigma_n)^2 + (\omega_n\eta_n)^2}}. \tag{4.26}$$

Let α denote an adherence point of the sequence $(\eta_n\omega_n)_{n\in\mathbb{N}}$. We consider the cases $\alpha \in [-\infty, +\infty] \setminus \{0\}$ and $\alpha = 0$.

Suppose first that $\alpha \in [-\infty, +\infty] \setminus \{0\}$. Then $(\omega_n)_{n\in\mathbb{N}}$ is unbounded since $\eta_n \to 0$ and we can restrict to a subsequence (σ_n, ω_n) such that $\omega_n \to +\infty$ (by symmetry, the case $\omega_n \to -\infty$ is analogous). From (4.26),

$$\frac{s_n}{\sqrt{1+\eta_n s_n}} = \frac{\omega_n}{|\omega_n|}\frac{(o(1)+i)\sqrt{1+o(1)-i\omega_n\eta_n}}{\sqrt{o(1)+\eta_n^2}}, \tag{4.27}$$

where the $o(1)$ are real valued. Thus

$$\Re\left(\frac{s_n}{\sqrt{1+\eta_n s_n}}\right) = -\frac{\omega_n}{|\omega_n|}\cdot\sqrt{\frac{|\omega_n\eta_n|}{o(1)+\eta_n^2}}\cdot \tag{4.28}$$

$$\cdot\left[\Im\left(\sqrt{\frac{1+o(1)}{|\omega_n\eta_n|} - i\frac{\omega_n\eta_n}{|\omega_n\eta_n|}}\right) + o(1)\Re\left(\sqrt{\frac{1+o(1)}{|\omega_n\eta_n|} - i\frac{\omega_n\eta_n}{|\omega_n\eta_n|}}\right)\right] \tag{4.29}$$

We then have, for $\alpha \in \mathbb{R}\setminus\{0\}$,

$$\sqrt{\left(\sqrt{\frac{1+o(1)}{|\omega_n\eta_n|} - i\frac{\omega_n\eta_n}{|\omega_n\eta_n|}}\right)} \to \sqrt{\frac{1-i\alpha}{|\alpha|}}, \tag{4.30}$$

and for $\alpha = \pm\infty$,

$$\sqrt{\left(\sqrt{\frac{1+o(1)}{|\omega_n\eta_n|} - i\frac{\omega_n\eta_n}{|\omega_n\eta_n|}}\right)} \to \sqrt{\mp i}, \tag{4.31}$$

such that, since $\alpha \neq 0$,

$$\lim_{n\to+\infty} \Re\left(\left(\sqrt{\frac{1+o(1)}{|\omega_n\eta_n|} - i\frac{\omega_n\eta_n}{|\omega_n\eta_n|}}\right)\right) \in (0,+\infty), \tag{4.32}$$

$$\lim_{n\to+\infty} \Im\left(\left(\sqrt{\frac{1+o(1)}{|\omega_n\eta_n|} - i\frac{\omega_n\eta_n}{|\omega_n\eta_n|}}\right)\right) \in (-\infty,+\infty)\setminus\{0\}. \tag{4.33}$$

From (4.28) and (4.32) we deduce that

$$\Re\left(\frac{s_n}{\sqrt{1+\eta_n s_n}}\right) \to -\infty, \quad \text{or} \quad \Re\left(\frac{s_n}{\sqrt{1+\eta_n s_n}}\right) \to +\infty, \tag{4.34}$$

which implies

$$\lim_{n\to+\infty} \left|e^{s_n/\sqrt{1+\eta_n s_n}} - e^{-s_n/\sqrt{1+\eta_n s_n}}\right| \to +\infty. \tag{4.35}$$

Since $\overline{\sigma} \in (-\infty,+\infty)$, we deduce $|e^{\sigma_n+i\omega_n}| \to e^{\overline{\sigma}} \in (0,+\infty)$ and $|\sqrt{1+\eta_n s_n}|^2 \geq 1/2$ for n sufficiently large. Hence using (4.35),

$$\lim_{n\to+\infty} \left|e^{s_n}\sqrt{1+\eta_n s_n}\right| \left|e^{s_n/\sqrt{1+\eta_n s_n}} - e^{-s_n/\sqrt{1+\eta_n s_n}}\right| = +\infty, \tag{4.36}$$

which contradicts (4.13).

Now we see the case $\alpha = 0$. This means that $\eta_n s_n \to 0$, and since $\eta_n\omega_n^2 \to +\infty$ by assumption, the sequence $(\omega_n)_{n\in\mathbb{N}}$ has to be unbounded and we can select a subsequence such that $\omega_n \to +\infty$ (resp. $-\infty$). Then, we have

$$\frac{s_n}{\sqrt{1+\eta_n s_n}} = s_n\left(1 - \frac{\eta_n s_n}{2} + o(\eta_n s_n)\right) = (\sigma_n + i\omega_n)\left(1 - \frac{\eta_n(\sigma_n+i\omega_n)}{2} + o(\eta_n s_n)\right), \tag{4.37}$$

where the function $o(\eta_n s_n)$ can be complex valued. Hence,

$$\Re\left(\frac{s_n}{\sqrt{1+\eta_n s_n}}\right) = (1+o(1))\sigma_n + \frac{\eta_n\omega_n^2}{2}(1+o(1)) \to +\infty,$$

where we used that $\sigma_n/\omega_n \to 0$. This means that we have again (4.35) and (4.36), which yields again a contradiction.

□

Lemma 4.7 (Lemma 2, [BCH25b]). *Let $(\eta_n)_{n\in\mathbb{N}}$ be a sequence of viscosities and $(s_n)_{n\in\mathbb{N}}$ an associated sequence of poles, with induced sequences $(\sigma_n)_{n\in\mathbb{N}}$ and $(\eta_n\omega_n^2)_{n\in\mathbb{N}}$. Let $\overline{\sigma}$ be an accumulation point of $(\sigma_n)_{n\in\mathbb{N}}$. Then*

$$\overline{\sigma} \leq -\ln(2). \tag{4.38}$$

Proof. By Lemma 4.6, $\overline{\sigma} \in [-\infty, +\infty)$. If $\overline{\sigma} = -\infty$, the lemma holds trivially, so we focus on the case $\overline{\sigma} \in \mathbb{R}$. By the second part of Theorem 4.6,

$$\lim_{n\to\infty} \sigma_n = \overline{\sigma} \in \mathbb{R}, \qquad \lim_{n\to\infty} \eta_n \omega_n^2 = 2\theta \in [0, +\infty).$$

In particular $\eta_n \omega_n \to 0$ as $n \to \infty$ by algebra of limits. We further claim that

$$\theta_n = \sqrt{1 + \eta_n s_n} = 1 + o(1), \tag{4.39}$$

$$\frac{s_n}{\theta_n} = s_n \left(1 - \frac{\eta_n s_n}{2}\right) + o(1) = \overline{\sigma} + \theta + i\omega_n + o(1), \tag{4.40}$$

$$e^{s_n} = e^{\overline{\sigma}} e^{i\omega_n} + o(1). \tag{4.41}$$

Indeed, the first two follow from the Taylor expansions around $z = 0$, for $a > 0$,

$$\sqrt{1+z} = 1 + \frac{z}{2} + O(z^2), \qquad \frac{z}{\sqrt{1+az}} = z - \frac{a}{2}z^2 + O(z^3).$$

Substituting $z = \eta_n s_n$, we obtain

$$\theta_n = 1 + o(1),$$

and

$$\frac{s_n}{\theta_n} = s_n \left(1 - \frac{\eta_n s_n}{2}\right) + o(1).$$

Writing $s_n = \sigma_n + i\omega_n$,

$$\frac{s_n}{\theta_n} = (\sigma_n + i\omega_n)\left(1 - \frac{\eta_n \sigma_n}{2} - \frac{i\eta_n \omega_n}{2}\right) \tag{4.42}$$

$$= \sigma_n - \frac{\eta_n \sigma_n^2}{2} + i\omega_n - i\eta_n \omega_n \sigma_n + \frac{\eta_n \omega_n^2}{2} \tag{4.43}$$

$$= \overline{\sigma} + i\omega_n + \theta + o(1), \tag{4.44}$$

where we took the limit $\eta_n \omega_n \to 0$. The third expression is immediate.

Thanks to the characteristic equation (4.13) and the expressions above, we obtain

$$e^{\overline{\sigma}} e^{i\omega_n} \left(e^{\overline{\sigma}+\theta+i\omega_n} - e^{-\overline{\sigma}-\theta-i\omega_n}\right) + 1 = o(1). \tag{4.45}$$

Looking at the imaginary part of this equation, we obtain

$$e^{2\overline{\sigma}+\theta} \sin(2\omega_n) = o(1),$$

from which $\sin(2\omega_n) = o(1)$. Hence $\cos^2(2\omega_n) = 1 - o(1)$ and, after extracting a subsequence,

$$e^{2i\omega_n} = 1 + o(1), \quad \text{or} \quad e^{2i\omega_n} = -1 + o(1).$$

We claim the first case is impossible: if it held, (4.45) would give

$$(e^{2\overline{\sigma}+\theta} - e^{-\theta}) + 1 = 0,$$

which implies $e^{2\overline{\sigma}} = e^{-2\theta} - e^{-\theta}$. Since $\theta \geq 0$, we would get $e^{2\overline{\sigma}} \leq 0$, which is impossible. So we must be in the second case, and by the same argument,

$$e^{2\overline{\sigma}} = e^{-\theta} - e^{-2\theta}.$$

We can solve a one-variable maximization problem associated with the right-hand side to find that the maximum of this function is attained at $\theta = \ln(2)$. Hence

$$\max\{e^{-\theta} - e^{-2\theta} : \theta \in [0, +\infty)\} = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}.$$

From this we obtain

$$\overline{\sigma} \leq \frac{1}{2}\ln\left(\frac{1}{4}\right) = -\ln(2),$$

which proves the lemma. □

We can now begin the proof of Theorem 4.3.

Proof of Theorem 4.3. We first show the lower bound. Denote $\sigma = \Re(s)$ and $\omega = \Im(s)$ and introduce the change of variables

$$\gamma^2 = \eta, \qquad \varphi = \omega\gamma, \quad \gamma \in [0, +\infty).$$

With this, the characteristic equation (4.13) can be rewritten as

$$\mathcal{G}(\eta, s) = e^{\sigma + i\varphi/\gamma}\tilde{\theta}\left(e^{(\sigma+i\varphi/\gamma)/\tilde{\theta}} - e^{-(\sigma+i\varphi/\gamma)/\tilde{\theta}}\right) + 1 = \tilde{\mathcal{G}}(\gamma, \sigma, \varphi), \tag{4.46}$$

where we write

$$\tilde{\theta} = \sqrt{1 + \gamma^2\sigma + i\gamma\varphi}.$$

Consider $\mathcal{B} \in C^\infty(\mathbb{R}; \mathbb{C})$ defined by

$$\mathcal{B}(\varphi) := -e^{-\varphi^2/2} + 1, \tag{4.47}$$

and $\mathcal{A} : (0, +\infty) \times \mathbb{R} \times \mathbb{R} \to \mathbb{C}$ defined by

$$\mathcal{A}(\gamma, \sigma, \varphi) := \left(\tilde{\mathcal{G}}(\gamma, \sigma, \varphi) - \mathcal{B}(\varphi)\right) e^{-2i\varphi/\gamma}.$$

We claim, from the three definitions above, that there exists γ_0 such that $\mathcal{A}$ extends continuously to $\gamma = 0$, i.e., with domain

$$[0, \gamma_0) \times (-2\ln(2), +\infty) \times \mathbb{R}.$$

Indeed, note that in the second variable we only restrict, which is a continuous operation. For the first variable, note that there exists $\gamma_0 > 0$ such that, for every $(\gamma, \sigma, \varphi) \in [0, \gamma_0) \times (-2\ln(2), +\infty) \times \mathbb{R}$,

$$\Re(\tilde{\theta}^2) = \Re(1 + \gamma^2\sigma + i\gamma\varphi) > 0,$$

so the map $(\gamma, \sigma, \varphi) \mapsto 1/\sqrt{1 + \gamma^2\sigma + i\gamma\varphi}$ is continuous, since the square root is holomorphic on $\mathbb{C} \setminus \mathbb{R}_-$. This shows the function is continuous on $(0, \gamma_0) \times (-2\ln(2), +\infty) \times \mathbb{R}$, so it remains to study its behavior as $\gamma \to 0^+$. Note that we can write

$$\mathcal{A}(\gamma, \sigma, \varphi) = e^{\sigma(1+1/\tilde{\theta})}e^{-i(\varphi/\gamma)(1-1/\tilde{\theta})}\tilde{\theta} \tag{4.48}$$

$$- e^{\sigma(1-1/\tilde{\theta})}e^{-i(\varphi/\gamma)(1+1/\tilde{\theta})}\tilde{\theta} + e^{-\varphi^2/2}e^{-2i\varphi/\gamma}. \tag{4.49}$$

Note that, in the first line of (4.48), the limit as $\gamma \to 0^+$ is $e^{2\sigma+\varphi^2/2}$. Indeed, it suffices to show that

$$\sigma(1 + 1/\tilde{\theta}) - i(\varphi/\gamma)(1 - 1/\tilde{\theta}) \to 2\sigma + \varphi^2/2.$$

This again follows from Taylor expansions around $z = 0$,

$$\frac{1}{\sqrt{1+z}} = 1 - \frac{z}{2} + \frac{3z^2}{2} + O(z^3).$$

Substituting $z = \eta s = \gamma^2\sigma + i\gamma\varphi$, we have

$$\begin{aligned}\sigma(1+1/\tilde{\theta}) - i(\varphi/\gamma)(1-1/\tilde{\theta}) &= 2\sigma - (\gamma^2\sigma + i\gamma\varphi)\left(\frac{\sigma}{2} + \frac{i\varphi}{2\gamma}\right) + (\gamma^2\sigma + i\gamma\varphi)^2\left(\frac{3\sigma}{2} + \frac{3\varphi}{2\gamma}\right) \\ &\quad + o(\gamma) \\ &= 2\sigma - \frac{\gamma^2\sigma^2}{2} - i\varphi\gamma\sigma + \frac{\varphi^2}{2} + o(1) \to 2\sigma + \frac{\varphi^2}{2},\end{aligned}$$

which proves the claim. On the other hand, the second line of (4.48) converges to 0 as $\gamma \to 0^+$. Indeed, by the Taylor expansion around $z = 0$,

$$\begin{aligned}\sigma(1-1/\tilde{\theta}) - i(\varphi/\gamma)(1+1/\tilde{\theta}) &= -\frac{i\varphi}{\gamma}\left(2 - \frac{\gamma^2\sigma}{2} - \frac{i\gamma\varphi}{2} + (\gamma^2\sigma + i\gamma\varphi)^2\left(\frac{3}{2} + \frac{3}{i\varphi}\gamma\sigma\right)\right) \\ &= -\frac{i\varphi}{\gamma}\left(2 - \frac{(\gamma^2\sigma + i\gamma\varphi)}{2} + o((\eta s)^2)\right),\end{aligned}$$

so that

$$\begin{aligned}&-e^{\sigma(1-1/\tilde{\theta})}e^{-i(\varphi/\gamma)(1+1/\tilde{\theta})}\tilde{\theta} + e^{-\varphi^2/2}e^{-2\varphi i/\gamma} \\ &\qquad = e^{-\varphi^2/2}e^{-2i\varphi/\gamma} - (1+o(1))e^{-i(\varphi/\gamma)(2-(\gamma^2\sigma+i\gamma\varphi)/2+o((\gamma^2\sigma+i\gamma\varphi)))} \\ &\qquad = e^{-\varphi^2/2}e^{-2i\varphi/\gamma} - (1+o(1))e^{-2i(\varphi/\gamma)-\varphi^2/2+o(1)} \\ &\qquad = \left(e^{-\varphi^2/2}e^{-2i\varphi/\gamma}\right)o(1),\end{aligned}$$

which proves the second part. Note that the limit above does not depend on σ or φ, which completes the extension to $[0, \gamma_0)$:

$$\mathcal{A}(0, \sigma, \varphi) = e^{2\sigma + \varphi^2/2}, \qquad \forall (\sigma, \varphi) \in (-2\ln(2), +\infty) \times \mathbb{R}. \tag{4.50}$$

We can use this value together with the definition of $\mathcal{B}$ to note that

$$\mathcal{A}(0, -\ln(2), \sqrt{2\ln(2)}) = \frac{1}{2}, \qquad \mathcal{B}(\sqrt{2\ln(2)}) = \frac{1}{2},$$

and hence

$$-\mathcal{A}(0, -\ln(2), \sqrt{2\ln(2)}) + \mathcal{B}(\sqrt{2\ln(2)}) = 0.$$

That is, we found a pole. To prove the lower bound of the theorem, the idea is to find a solution $(\tilde{\sigma}, \tilde{\varphi})$ of the characteristic equation,

$$\tilde{\mathcal{G}}(\gamma, \tilde{\sigma}, \tilde{\varphi}) = e^{2i\tilde{\varphi}/\gamma}\mathcal{A}(\gamma, \tilde{\sigma}, \tilde{\varphi}) + \mathcal{B}(\tilde{\varphi}) = 0, \tag{4.51}$$

such that $\tilde{\sigma}$ (resp. $\tilde{\varphi}$) is close to $-\ln(2)$ (resp. $\sqrt{2\ln(2)}$). In particular, this will imply that $e^{2i\tilde{\varphi}/\gamma}$ is close to -1. For this we use topological degree theory (Section 2.3).

Let $\gamma \in (0, \gamma_0)$. Given $\delta \in (0, \ln(2))$ as in the theorem, consider the open subset of $\mathbb{R}^2$,

$$\Omega^\gamma = (-\ln(2) - \delta, -\ln(2) + \delta) \times (k(\gamma)\pi\gamma, (k(\gamma)+1)\pi\gamma),$$

where the function $k(\gamma)$ is defined by

$$k(\gamma) := \left\lfloor \frac{\sqrt{2\ln(2)}}{\pi\gamma} \right\rfloor,$$

and a function $\phi_\gamma : \overline{\Omega^\gamma} \to \mathbb{C}$ given by

$$\phi_\gamma : (\sigma, \varphi) \mapsto e^{2i\varphi/\gamma}\mathcal{A}(\gamma, \sigma, \varphi) + \mathcal{B}(\varphi) \in \mathbb{C}. \tag{4.52}$$

With these definitions we can prove the following.

Lemma 4.8 (Lemma 3, [BCH25b]). *There exists $\gamma_1 > 0$ such that, for every $\gamma \in (0, \gamma_1]$,*

$$\deg(\phi_\gamma, \Omega^\gamma, 0) = 1.$$

We will prove this after the main argument. The idea behind the topological degree is that, given an open set and a function defined on it (here Ω^γ and ϕ_γ), we can study how many times the image of ϕ_γ winds around 0. That is, the lemma tells us that exactly one loop around 0 is completed.

Using Lemma 4.8, and thanks to Proposition 2.17 ([Cor07, Prop. B.10]), since the degree is nonzero, there exists at least one point in Ω^γ whose image is 0. More precisely, for every $\delta \in (0, \ln(2))$ and $\gamma \in (0, \gamma_1]$, there exists $(\tilde{\sigma}, \tilde{\varphi}) \in \Omega^\gamma$ such that $\phi_\gamma(\tilde{\sigma}, \tilde{\varphi}) = 0$, i.e., the characteristic equation (4.51) holds. Writing $\tilde{s} = \tilde{\sigma} + i\frac{\tilde{\varphi}}{\gamma}$, it follows from the definition of ϕ_γ that

$$\mathcal{G}(\gamma^2, \tilde{s}) = 0,$$

hence, since $0 < \gamma \leq \gamma_1$, for every $\eta \in (0, \gamma_1^2]$ there exists $\tilde{s} \in \mathcal{S}_\eta$ with $\Re(\tilde{s}) \in (-\ln(2) - \delta, -\ln(2) + \delta)$. Taking the supremum, we get

$$-\ln(2) - \delta < \Re(\tilde{s}) = \tilde{\sigma} \leq \sigma_\eta,$$

by the definition of the maximal spectral abscissa. This proves the lower bound of the theorem.

For the upper bound we cannot simply take the supremum, so we argue differently. Recall from the proof of Theorem 3.2 in the two-viscous-terms case that what we need, a priori, is a bound analogous to Lemma 3.6. In the present proof this is provided by Lemma 4.7, so we can replicate the short contradiction argument of Theorem 3.2 to obtain that there exists $\eta_1 > 0$ such that

$$\sigma_\eta < -\ln(2) + \delta, \qquad \forall \eta \in (0, \eta_1].$$

Finally, Theorem 4.3 is obtained using $\eta^* = \min\{\gamma_1^2, \eta_1\}$. □

Proof of Lemma 4.8. Recall that $\delta \in (0, \ln(2))$ and $\gamma \in (0, \gamma_0]$. Since A and B are continuous on $[0, \gamma) \times (-\ln(2) - \delta, -\ln(2) + \delta) \times \mathbb{R}$, we have

$$\phi^\gamma(\sigma, k(\gamma)\pi\gamma) = 2e^{2\sigma} + \frac{1}{2} + o(1), \tag{4.53}$$

$$\phi^\gamma(-\ln(2) + \delta, \varphi) = \frac{1}{2}\left(e^{2i\varphi/\gamma}e^{2\delta} + 1\right) + o(1), \tag{4.54}$$

$$\phi^\gamma(\sigma, (k(\gamma) + 1)\pi\gamma) = 2e^{2\sigma} + \frac{1}{2} + o(1), \tag{4.55}$$

$$\phi^\gamma(-\ln(2) - \delta, \varphi) = \frac{1}{2}\left(e^{2i\varphi/\gamma}e^{-2\delta} + 1\right) + o(1), \tag{4.56}$$

where $o(1)$ refers to functions that then to 0 in the C^0-norm when $\gamma \to 0^+$. These estimates are direct consequences of (4.47), (4.50) and (4.52). We will focus on the case where (σ, φ) follows the boundary of the rectangle $(-\ln(2) - \delta, -\ln(2) + \delta) \times (k(\gamma)\pi\gamma, (k(\gamma) + 1)\pi\gamma)$ one has

1. On the left and right boundaries, the function $2e^{2\sigma} + 1/2$ remains on the half real line $(1/2, +\infty)$.

2. On the upper boundary $\sigma = -\ln(2) + \delta$, as φ/γ increases from $k(\gamma)\pi$ to $(k(\gamma)+1)\pi\gamma$, the function $(1/2)(e^{2i\varphi/\gamma}e^{2\delta}+1)$ never meets 0 but describes a closed curve independent of $\gamma > 0$ which encircles 0 exactly once and in the clockwise direction since $2\varphi/\gamma - 2k(\gamma)\pi$ increases from 0 to 2π and $e^{2\delta} \in (1, +\infty)$.

3. On the lower boundary $\sigma = -\ln(2) - \delta$, the function $(1/2)(e^{2i\varphi/\gamma}e^{-2\delta}+1)$ does not pass through 0 but describes a curve independent of $\gamma > 0$ which remains in the strict right-hand side of the complex plane $\{z \in \mathbb{C} \mid \Re(z) > 0\}$ since $|e^{2i\varphi/\gamma}e^{-2\delta}| < 1$.

Hence, using (4.53), there exists $\gamma_1 \in (0, \gamma_0)$ such that, for every $\gamma \in (0, \gamma_1)$, when (σ, φ) follows the boundary of the rectangle Ω^γ in the clockwise direction, the whole curve described by ϕ^γ does not pass through the origin 0, but encircles the origin exactly once and in the clockwise direction, which means that (4.52) holds. This completes the proof of Lemma 4.8.

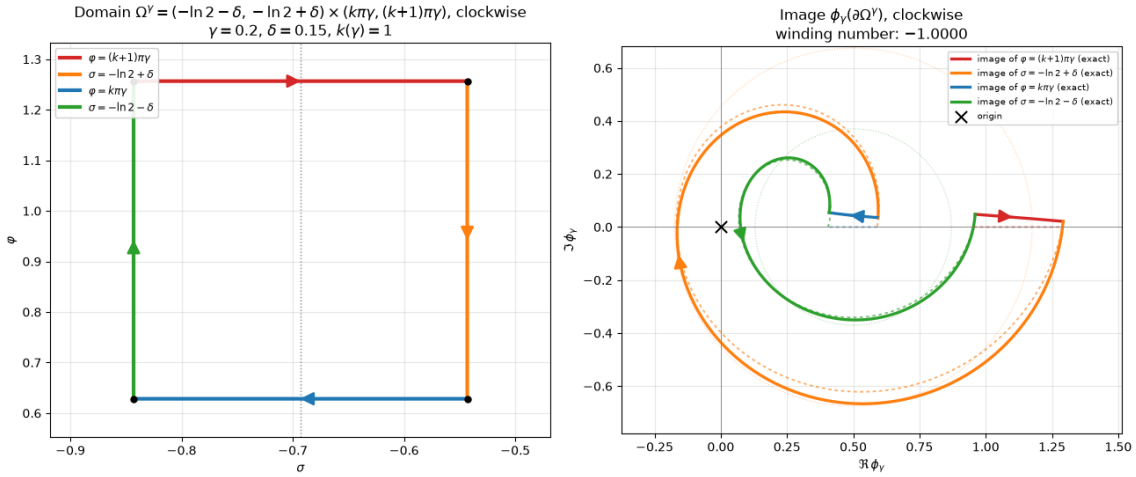


Figure 4.2: Plane region

□

Remark 4.9. From [Cor07, Prop. B.10] it follows that there is at least one point that is a preimage of 0. Shrinking $\gamma_1 > 0$ further if necessary, we may restrict to the case where this preimage is unique, which is what we use in the proof. This occurs since for every $\eta > 0$, there exists $\delta_\eta > 0$ such that the function $s \mapsto \mathcal{G}(\eta, s)$ is holomorphic on $\{z \in \mathbb{C} \mid \Re(z) > \delta_\eta\}$. In particular, the choice of the interval for σ is made so that the maximal spectral abscissa does not escape from the region where $\Re(s) = \sigma$ is negative, thus proving stabilization. This further shows that, for

$$\tilde{\omega} \in \Im(\tilde{s}) \in (k(\gamma)\pi, (k(\gamma)+1)\pi),$$

we have $\lim_{\eta \to 0^+} \eta\tilde{\omega}^2 = 2\ln(2)$.

Remark 4.10 (Important)**.** The topological-degree argument can be carried out in greater generality. We record the required hypotheses. Consider $\mathcal{A} : [0, +\infty) \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}^2$ and $\mathcal{B} : (0, +\infty) \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}^2$ continuous, and assume there exist $\sigma_* \in \mathbb{R}$, $\delta_* \in (0, +\infty)$ and $\varphi_* \in (0, +\infty)$ such that

$$\mathcal{A}(0, \sigma_*, \varphi_*) = \mathcal{B}(0, \sigma_*, \varphi_*) \neq 0,$$

and the map

$$\sigma \mapsto |\mathcal{A}(0, \sigma_*, \varphi_*)| - |\mathcal{B}(0, \sigma_*, \varphi_*)|$$

is strictly monotone on $[\sigma_* - \delta_*, \sigma_* + \delta_*]$. Then one can prove that, for every $\delta_0 \in (0, \delta_*)$, there exists $\gamma_0 > 0$ such that for every $\gamma \in (0, \gamma_0)$ there exists $(\tilde{\sigma}, \tilde{\varphi}) \in \mathbb{R}^2$ such that

$$\tilde{\mathcal{G}}(\gamma, \tilde{\sigma}, \tilde{\varphi}) = e^{2i\tilde{\varphi}/\gamma}\mathcal{A}(\gamma, \tilde{\sigma}, \tilde{\varphi}) + \mathcal{B}(\gamma, \tilde{\sigma}, \tilde{\varphi}) = 0,$$

for

$$(\tilde{\sigma}, \tilde{\varphi}) \in (\sigma_* - \delta_0, \sigma_* + \delta_0) \times (k(\gamma)\pi\gamma, (k(\gamma) + 1)\pi\gamma),$$

where

$$k(\gamma) := \left\lfloor \frac{\varphi_*}{\pi\gamma} \right\rfloor.$$

With this generalization one can prove Conjecture 1 of Chapter 3.

Remark 4.11. Recall that throughout this chapter we normalized the constants $c = L = 1$, so that we obtain a stability margin of $\ln(2)$. If we use arbitrary c, L, this stability margin becomes $\frac{c}{L}\ln(2)$.

4.5 Robustness Analysis

Consider a perturbation of the fluid flow, i.e., we write $c^2 = 1 + \varepsilon$, obtaining the system

$$\begin{cases} \partial_t h(t,x) + \partial_x q(t,x) = 0, \\ \partial_t q(t,x) + (1+\varepsilon)\partial_x h(t,x) - \eta\partial_{xx}^2 q(t,x) = 0, \\ q(t,0) = U(t), \\ q(t,1) = 0, \\ Y(t) = h(t,1). \end{cases} \tag{4.57}$$

We consider the same feedback law to close the system, $U(t) = -\frac{1}{2}Y(t-1)$. Following the same calculations from Section 4.4, we get the following modified characteristic equation in the frequency domain,

$$\mathcal{G}_\varepsilon(\eta, s) = e^s\sqrt{1+\varepsilon+\eta s}\left(e^{s/\sqrt{1+\varepsilon+\eta s}} - e^{-s/\sqrt{1+\varepsilon+\eta s}}\right) + 1 = 0. \tag{4.58}$$

As in the previous chapter, we introduce the spectrum and maximal spectral abscissa of the perturbed problem,

$$\begin{aligned} \mathcal{S}_{\eta,\varepsilon} &= \{s \in \mathbb{C} : s \text{ solves } (4.58)\}, \\ \sigma_{\eta,\varepsilon} &:= \sup\{\Re(s) : \ s \in \mathcal{S}_{\eta,\varepsilon}\}. \end{aligned}$$

Recall that, by definition,

$$\mathcal{S}_{\eta,0} = \mathcal{S}_\eta, \qquad \sigma_{\eta,0} = \sigma_\eta.$$

Then we have the following robustness theorem.

Theorem 4.12 (Theorem 2, [BCH25b])**.** *Let $\delta > 0$ and $\eta > 0$ be as in Theorem 4.3, i.e.,*

$$\sigma_{\eta,0} \le -\ln(2) + \delta.$$

Then there exists $\varepsilon_1 > 0$ such that, for every $\varepsilon \in (-\varepsilon_1, \varepsilon_1)$, the maximal spectral abscissa $\sigma_{\eta,\varepsilon}$ satisfies

$$\sigma_{\eta,\varepsilon} \le -\ln(2) + 2\delta.$$

The proof of the theorem is, in essence, the same as that of Theorem 3.7.

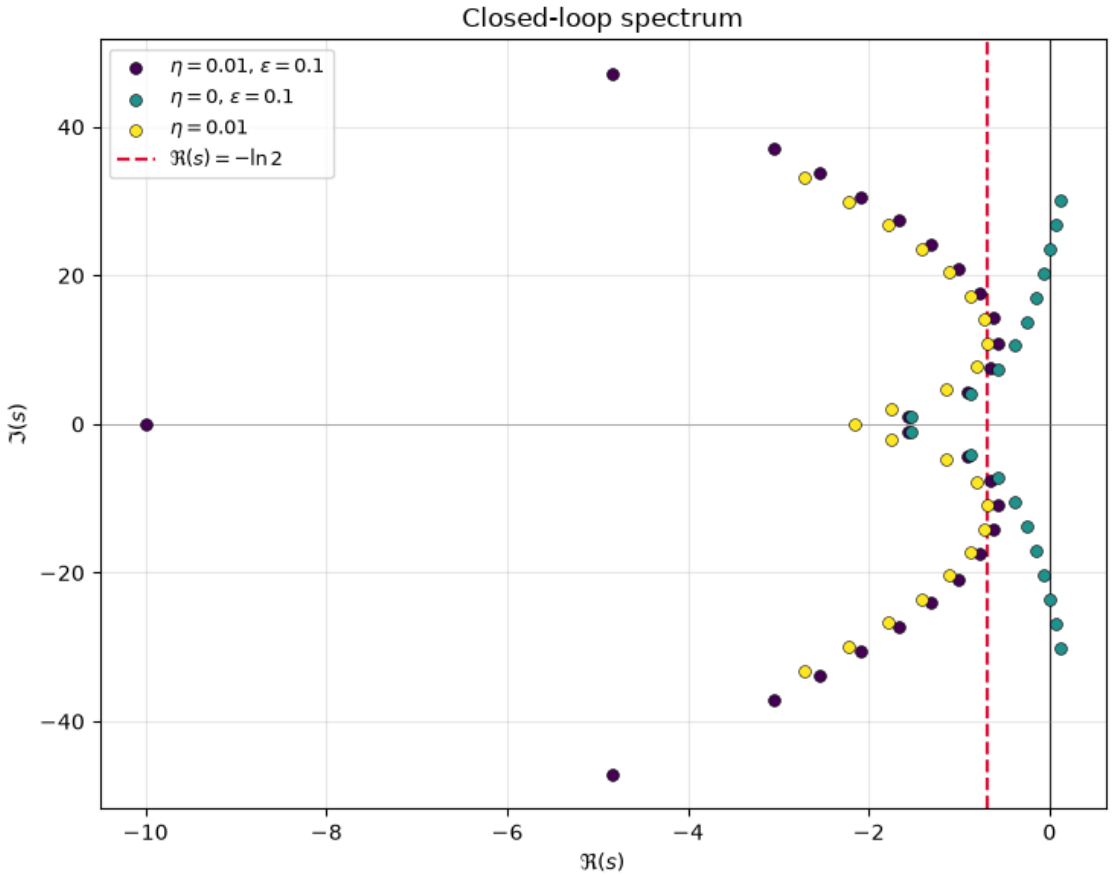


Figure 4.3: Closed-loop spectrum

Chapter 5

Stability and robustness with respect to fourth-order perturbations for a hyperbolic system with boundary coupling

5.1 Introduction

The results of Chapter 3 and Chapter 4 show that a small second-order diffusion term is enough to restore the robustness of a delayed output-feedback law that stabilizes a hyperbolic system with anti-located control and observation. In both cases, the argument rests on an explicit description of the roots of a quadratic characteristic equation, obtained through the auxiliary variable $z = \sqrt{1 + 4\eta s}$. This naturally raises a structural question: does the same robustness phenomenon persist if the second-order diffusion term $\eta\,\partial_x^2$ is replaced by a fourth-order dissipative term $\eta\,\partial_x^4$, of the kind that arises, for instance, in beam and thin-film models?

On the one hand, a fourth-order operator in x requires four boundary conditions per channel instead of two, and choosing them consistently with the underlying hyperbolic structure already demands some care. On the other, and more importantly, the explicit change of variable that reduces the second-order characteristic equation to a solvable quadratic no longer applies. The resulting characteristic equation is now a genuine quartic, whose four roots have no closed-form expression amenable to the same kind of direct manipulation. Consequently, the strategy of Chapter 4 cannot be replicated, and has to be adapted at the level of the asymptotic analysis of the roots themselves.

The approach we follow in this chapter is to replace the explicit root formulas by an asymptotic classification of the four characteristic roots as $\eta \to 0^+$, precise enough to produce a first-order expansion of the transfer function together with an explicit, controllable error term. This expansion is then plugged into the same topological degree argument used in Chapter 4 to obtain a lower bound on the maximal spectral abscissa. Concretely, the chapter proceeds through the following steps:

1. Construct the single-channel transfer function $f_\eta(s)$ of the fourth-order regularized transport equation, and identify the resulting closed-loop characteristic equation.
2. Classify the four roots of the characteristic quartic equation according to their asymptotic behavior as $\eta \to 0^+$, separating a finite branch from three exploding branches.
3. Use this classification to derive an asymptotic expansion of $f_\eta(s)$, first on compact subsets of $\mathbb{C}$ and then along the frequency scale relevant to the stability problem.
4. Prove asymptotic lemmas analogous to Theorem 3.6, Theorem 4.7 and Theorem 4.8, adapted to the fourth-order setting.
5. Combine these estimates with topological degree theory, as in Chapter 4, to obtain a lower bound for the maximal spectral abscissa. The upper bound is also proven

following the exposure of Chapter 3, and hence the robustness result of Theorem 5.5.

We begin, in the next section, by writing down the open-loop system and the associated transfer function, so that the difficulty caused by the loss of an explicit root formula becomes precise.

5.2 The open-loop control system with fourth order perturbation

We consider two identical transport channels on the spatial domain $x \in (0,1)$, each regularized by a fourth-order dissipative term, and coupled in a loop through a boundary feedback law that uses an internal delay predictor. Precisely, for $\eta > 0$ we study the system

$$\begin{cases} \partial_t y_1(t,x) + \partial_x y_1(t,x) + \eta \partial_x^4 y_1(t,x) = 0, & t > 0,\ x \in (0,1), \\ \partial_t y_2(t,x) + \partial_x y_2(t,x) + \eta \partial_x^4 y_2(t,x) = 0, & t > 0,\ x \in (0,1), \\ y_1(t,0) = y_2(t,1) + U(t), \\ y_2(t,0) = y_1(t,1), \\ \partial_x^2 y_i(t,0) = \partial_x^2 y_i(t,1) = \partial_x^3 y_i(t,1) = 0, \quad i = 1,2, \\ Y(t) = y_1(t,1). \end{cases} \tag{5.1}$$

Here y_1, y_2 are the two biharmonically regularized transport states, $U(t)$ denotes the feedback law and $Y(t)$ the output measurement of the system. This system should be compared with its second-order counterpart Equation (3.7).

Remark 5.1. Since (5.1) involves a fourth-order differential operator in x, each of y_1, y_2 requires four boundary conditions instead of two. Consistently with the original hyperbolic (first-order transport) behavior of the system, we impose a no-flux-type choice on the two extra boundary conditions, namely

$$\partial_x^2 y_i(t,0) = 0, \quad \partial_x^2 y_i(t,1) = 0, \quad \partial_x^3 y_i(t,1) = 0.$$

This particular choice is guided both by numerical implementations and by the requirement that the transfer function converge to the inviscid one, $f_\eta(s) \to e^{-s}$, as $\eta \to 0^+$. This convergence is established in Section 5.5.

The boundary coupling in (5.1) still yields a plant-type interconnection, so the block diagram of Figure 3.2 continues to describe the situation once the single-channel transfer function is known. To compute it, we work in the Laplace domain: fixing $i \in \{1,2\}$, consider the boundary value problem satisfied by $y_i(s,\cdot)$,

$$s y_i(s,x) + \partial_x y_i(s,x) + \eta \partial_x^4 y_i(s,x) = 0, \qquad x \in (0,1). \tag{5.2}$$

The general solution of (5.2) takes the form

$$y_i(s,x) = \sum_{k=1}^{4} A_k(s) e^{\lambda_k(s) x}, \tag{5.3}$$

where $\lambda_1(s), \dots, \lambda_4(s)$ are the four roots of the characteristic quartic equation

$$\eta\lambda^4 + \lambda + s = 0. \tag{5.4}$$

Unlike its second-order counterpart, (5.4) is not biquadratic, so no expression as (3.9) is available for $\lambda_k(s)$, $k \in \{1, \dots, 4\}$. In particular, the change of variable that produced the closed-form characteristic equation (3.14) in the second-order case has no fourth-order analogue, and an alternative route toward the transfer function must be found.

One could, in principle, invoke Ferrari's method to obtain $\lambda_k(s)$ explicitly. The resulting expressions, however, involve three nested square roots and are consequently unwieldy for our purposes. For instance, the four solutions take the form

$$\lambda_{\varepsilon,\pm}(s) = \frac{1}{2}\left[\varepsilon\sqrt{2M} \pm \sqrt{-2M - \frac{\varepsilon}{\eta}\sqrt{\frac{2}{M}}}\right], \quad \varepsilon \in \{-1, 1\},$$

where

$$M = \sqrt[3]{\frac{1}{16\eta^2} + \sqrt{\frac{1}{256\eta^4} - \frac{s^3}{27\eta^3}}} + \frac{s}{3\eta\sqrt[3]{\frac{1}{16\eta^2} + \sqrt{\frac{1}{256\eta^4} - \frac{s^3}{27\eta^3}}}}.$$

Rather than working directly with these expressions, we will instead study an asymptotic description of the $\lambda_k(s)$ as $\eta \to 0^+$, developed in Section 5.4. For the moment, we only need to record how the roots enter the boundary conditions. Imposing the four boundary conditions from (5.1) on $y_i(s,x) = \sum_{k=1}^{4} A_k(s)e^{\lambda_k(s)x}$ produces the linear system $M(s)\mathbf{A} = \mathbf{b}$, $A = (A_1, \dots, A_4)^\top$, with

$$M(s) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 & \lambda_4^2 \\ \lambda_1^2 e^{\lambda_1} & \lambda_2^2 e^{\lambda_2} & \lambda_3^2 e^{\lambda_3} & \lambda_4^2 e^{\lambda_4} \\ \lambda_1^3 e^{\lambda_1} & \lambda_2^3 e^{\lambda_2} & \lambda_3^3 e^{\lambda_3} & \lambda_4^3 e^{\lambda_4} \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} y_i(s,0) \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

This system is well posed whenever $M(s)$ is invertible, which holds for all s outside a discrete set. By Cramer's rule, the transfer function

$$f_\eta(s) = \frac{y_i(s,1)}{y_i(s,0)}, \tag{5.5}$$

is then well defined, as a ratio of two determinants built from $M(s)$. Unlike its second-order counterpart, however, $f_\eta(s)$ admits no closed form in terms of elementary functions of s. This is exactly what motivates the asymptotic study of f_η as $\eta \to 0^+$ carried out in Section 5.5.

Before turning to that asymptotic analysis, we first record the characteristic equation of the system, obtained, as in Section 4.3, directly from the boundary conditions rather than from the explicit form of f_η. Taking the Laplace transform in time of (5.1) gives

$$\begin{cases} sy_1(s,x) + \partial_x y_1(s,x) + \eta\partial_x^4 y_1(s,x) = 0, \\ sy_2(s,x) + \partial_x y_2(s,x) + \eta\partial_x^4 y_2(s,x) = 0, \\ y_1(s,0) = y_2(s,1) + U(s), \\ y_2(s,0) = y_1(s,1), \\ \partial_x^2 y_i(s,0) = \partial_x^2 y_i(s,1) = \partial_x^3 y_i(s,1) = 0, \quad i = 1,2, \\ Y(s) = y_1(s,1). \end{cases} \tag{5.6}$$

Combining (5.6) with the single-channel transfer function (5.5) applied to each of y_1, y_2, we obtain

$$\begin{cases} y_1(s,1) = f_\eta(s) y_1, \\ y_2(s,1) = f_\eta(s) y_2, \\ y_1(s,0) = y_2(s,1) + U(s), \\ y_2(s,0) = y_1(s,1), \\ \partial_x^2 y_i(s,0) = \partial_x^2 y_i(s,1) = \partial_x^3 y_i(s,1) = 0, \quad i = 1,2, \\ Y(s) = y_1(s,1). \end{cases} \tag{5.7}$$

The plant form of the system yields in this case the overall input-output transfer function of the open-loop system (5.1),

$$G(s) = \frac{Y(s)}{U(s)} = \frac{f_\eta(s)}{1 - f_\eta(s)^2}. \tag{5.8}$$

This means that the poles of the system are the roots of the characteristic equation

$$f_\eta(s)^2 - 1 = 0. \tag{5.9}$$

The open-loop spectrum for the system (5.1) is shown in Figure 5.2. Since we have a pole at the origin for every value of $\eta > 0$, we have the unstability of the open-loop system.

5.3 Closed-loop system

We now assume that the open-loop system (5.1) is closed with the output feedback controller

$$U(t) = -Y(t-1), \tag{5.10}$$

from which we obtain the closed-loop system

$$\begin{cases} y_1(s,1) = f_\eta(s) y_1, \\ y_2(s,1) = f_\eta(s) y_2, \\ \hat{y}_2(s,1) = e^{-s}\hat{y}_2(s,0) y_1(s,0) = y_2(s,1) - \hat{y}_2(s,1), \\ y_2(s,0) = y_1(s,1), \\ \partial_x^2 y_i(s,0) = \partial_x^2 y_i(s,1) = \partial_x^3 y_i(s,1) = 0, \quad i = 1,2, \\ \hat{y}_2(s,0) = y_1(s,1). \end{cases} \tag{5.11}$$

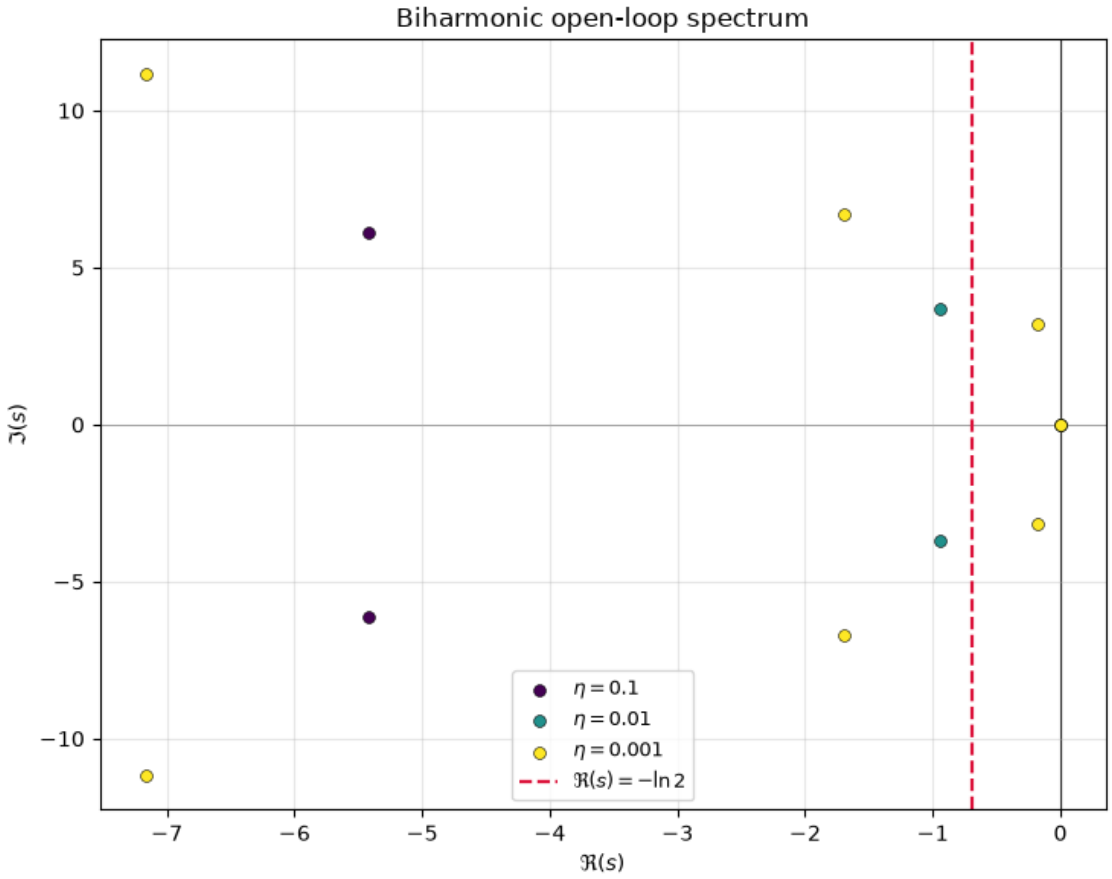


Figure 5.1: Open-loop spectrum of (5.1)

From these equations it follows that the characteristic equation of the closed-loop system becomes

$$f_\eta(s)^2 - f_\eta(s)e^{-s} - 1 = 0. \tag{5.12}$$

Indeed, the equation comes from the controller definition (5.10) and the overall input-output transfer function (5.8)

$$\frac{f_\eta(s)}{1 - f_\eta(s)^2} = G(s) = \frac{Y(s)}{U(s)} = \frac{1}{-e^{-s}}.$$

Remark 5.2. Besides having different transfer functions due to the fourth-order nature of system (5.1), qualitatively we are in the same situation as in Section 3.4.

The main tool to tackle the stabilization problem is still the study of the spectrum and maximal spectral abscissa of the closed-loop system. For the sake of completeness, we rewrite this definitions.

Definition 5.3. The spectrum and maximal spectral abscissa of the closed-loop system (5.1) are

$$\mathcal{S}_\eta := \{s \in \mathbb{C} : \ f_\eta(s)^2 - f_\eta(s)e^{-s} - 1 = 0\}, \qquad \sigma_\eta := \sup\{\Re(s) : \ s \in \mathcal{S}_\eta\}.$$

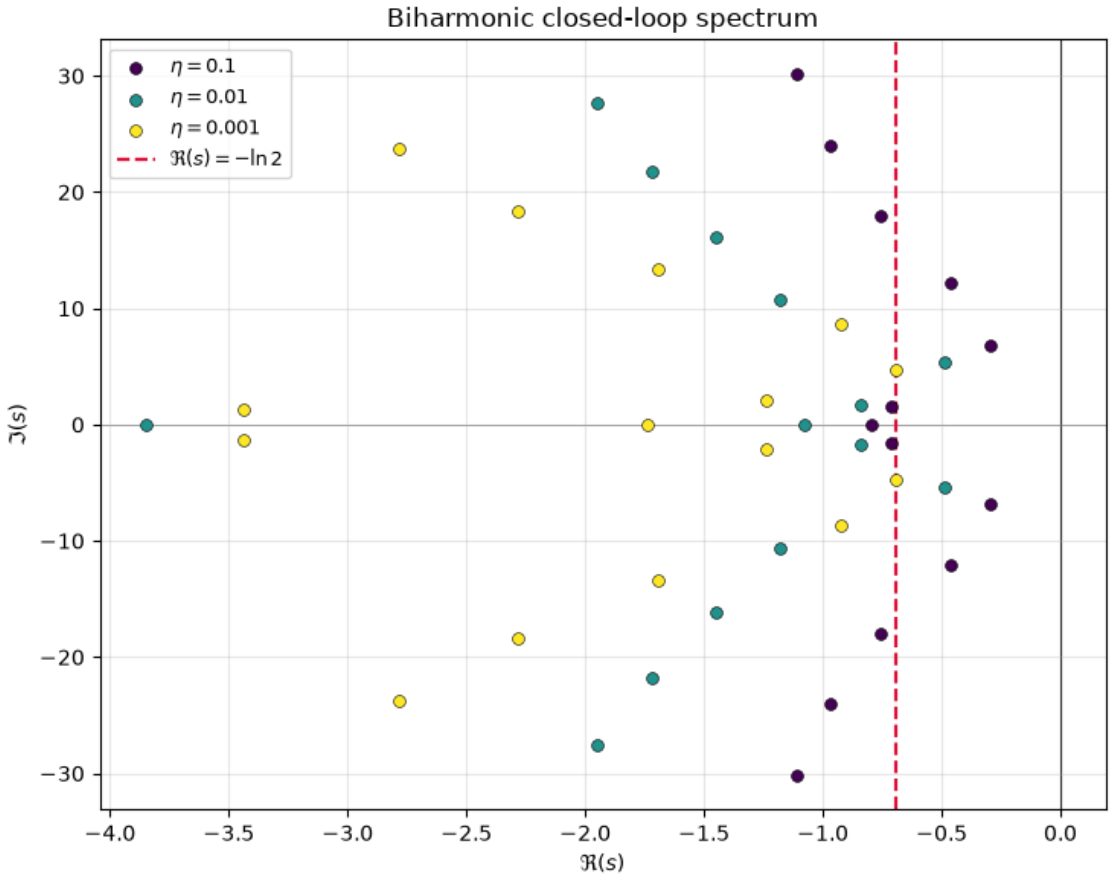


Figure 5.2: Closed-loop spectrum of (5.6)

Remark 5.4. By the spectral mapping theorem, the closed-loop semigroup is exponentially stable if and only if $\sigma_\eta < 0$. In this case, we don't know if (5.1) satisfies this desired property.

The main idea of the following sections it to prove the principal result of this chapter. This result is the fourth-order analogue of Theorem 3.2.

Theorem 5.5. *For every $\delta > 0$, there exists $\eta_1 > 0$ such that for all $\eta \in (0, \eta_1)$,*

$$\sigma_\eta \leq -\ln(2) + \delta.$$

In Section 5.4, we first characterize the roots of the characteristic equation (5.2) via Taylor expansions. Then, in Section 5.5 we develop an algebraic identity which will we helpful to rewrite a first order approximation of the transfer function f_η over a compact subset. Then we will extend this result to the important resonant scale given by the fourth-order perturbation.

Finally, in Section 5.6 we will show several lemmas which will prove Theorem 5.5.

5.4 Roots of the characteristic quartic

In this section our objective is to study the other three roots of the biharmonically modified transport equation. Once we identify these, we will develop an asymptotic description of

these branches in order to identify the principal one and get an expansion of the transfer function f_η.

Fix $s \in \mathbb{C}$. When we study the equation (5.2) we are adding 3 new solutions. For instance, if $\eta = 0$ we get the first root $\lambda = -s$. The other three roots of the quartic must diverge as $\eta \to 0^+$, which tells us that a boundary-layer problem is arising in the limit process. We make this precise.

Proposition 5.6. *For $s \in \mathbb{C}$ fixed, the four roots of (5.2) admit, as $\eta \to 0^+$, the classification*

$$\lambda_1(s) = -s + O(\eta), \tag{5.13}$$

$$\lambda_k(s) = \eta^{-1/3}\rho_k + O(1), \quad k \in \{2,3,4\}, \tag{5.14}$$

where ρ_2, ρ_3, ρ_4 are the three cube roots of -1 given by

$$\rho_2 = -1, \quad \rho_3 = e^{i\pi/3}, \quad \rho_4 = e^{-i\pi/3}.$$

In the following, we will say that $\lambda_1(s)$ is the finite branch and $\lambda_k(s)$ for $k \in \{2,3,4\}$ are the exploding branches.

Proof. For the finite branch, write $\lambda_1 = -s + \eta c_1 + O(\eta^2)$ and substitute into (5.4),

$$\eta(-s + \eta c_1)^4 + (-s + \eta c_1) + s = \eta s^4 + \eta c_1 + O(\eta^2) = 0,$$

using $(-s + \eta c_1)^4 = s^4 + O(\eta)$. Hence $c_1 = -s^4$ and we obtain

$$\lambda_1(s) = -s - \eta s^4 + O(\eta^2 s^7), \tag{5.15}$$

where the error term is obtained by continuing the expansion one step further. This shows (5.13).

For the exploding branches, set $\lambda_k(s) = \eta^{-1/3}(\rho_k + \mu)$ where $k \in \{2,3,4\}$ and $\mu \to 0$. Replacing in (5.4),

$$\eta^{-1/3}\Big[(\rho_k + \mu)^4 + (\rho_k + \mu)\Big] = -s.$$

At $\mu = 0$ we have $\rho_k^4 + \rho_k = \rho_k \cdot \rho_k^3 + \rho_k = \rho_k(-1) + \rho_k = 0$, confirming ρ_k solves the leading-order equation $w^4 + w = 0$, $w \neq 0$. Differentiating in μ at $\mu = 0$:

$$\frac{d}{d\mu}\Big[(\rho_k + \mu)^4 + (\rho_k + \mu)\Big]\Big|_{\mu=0} = 4\rho_k^3 + 1 = -4 + 1 = -3,$$

Hence $-3\mu \approx -s\eta^{1/3}$, i.e.

$$\mu = \frac{s}{3}\eta^{1/3} + O(\eta^{5/3}). \tag{5.16}$$

Going one step further to get the second term approximation, we obtain

$$\lambda_k(s) = \eta^{-1/3}\rho_k + \frac{s}{3} + O(\eta^{2/3}), \qquad k = 2,3,4. \tag{5.17}$$

This proves (5.14), with the sharper form (5.17) recorded for later use. □

Remark 5.7. Since $\Re(\rho_2) = -1 < 0$ and $\Re(\rho_3) = \Re(\rho_4) = \frac{1}{2} > 0$, the exploding branch λ_2 produces a boundary layer at $x = 0$ associated to the mode $e^{\lambda_2 x}$, which is negligible away from $x = 0$, while λ_3, λ_4 produce a two-dimensional boundary layer at $x = 1$. This is consistent with the boundary conditions imposed in (5.1). In the following we explain what we mean by boundary layer.

When we see system (5.1) as a perturbation of the former transport system (3.1), we are adding three new solutions with the necessary boundary conditions. From Proposition (5.6) we have that $|\lambda_k(s)| \to +\infty$ for $k \in \{2, 3, 4\}$ meaning that we expect these solutions to explode as $\eta \to 0^+$ by the asymptotic expansion obtained, which is the case since the modes $e^{\lambda_k(s)x}$ varies over an x lenght scale of order $\eta^{1/3} \to 0$ concentrated in a thin strip of width $O(\eta^{1/3})$, being adjacent to $x = 0$ or $x = 1$. This thin transition region which shrinks as $\eta \to 0^+$ is what we call a boundary layer phenomenon.

Remark 5.8. In the previous proposition we followed the exposition of [Kat66, Chapter 2, Section 2] to get the desired expressions for $\lambda_1, \dots, \lambda_4$. The idea is the following.

Consider the perturbed equation

$$G(\lambda, \eta) = \eta\lambda^4 + \lambda + s.$$

At $\eta = 0$ we know that $\overline{\lambda} = -s$ is a root, and in particular,

$$\frac{\partial G}{\partial \lambda}(\overline{\lambda}, 0) = \left[4\eta\lambda^3 + 1\right] |_{\lambda=-s,\eta=0} = 1 \neq 0.$$

The Implicit Function Theorem says that there is a unique branch $\lambda_1(\eta)$, which is analytic in η near 0, with $\lambda_1(0) = -s$. Furthermore, the same Theorem says that

$$\lambda_1'(0) = -\frac{\frac{\partial G}{\partial \eta}}{\frac{\partial G}{\partial \lambda}} |_{(-s,0)} = \frac{-\lambda^4}{1} |_{(-s,0)} = -s^4,$$

which gives $\lambda_1(s) = -s - \eta s^4 + O(\eta^2)$ as in (5.15).

Now we study $G(\lambda, \eta)$ to get the other 3 solutions. Replace $\lambda = \eta^{-1/3}\nu$ to get

$$G(\nu, \eta) = \eta^{-1/3}\nu^4 + \eta^{-1/3}\nu + s.$$

When we look at the solutions of the previous equation, we get $\nu^4 + \nu + \eta^{1/3}s = 0$. Taking $\eta \to 0$ yields

$$\nu^4 + \nu = \nu(\nu^3 + 1) = 0.$$

Here we have four roots given by $\nu_1 = 0$ and $\nu_k = \rho_k$ for $k \in \{2, 3, 4\}$ with $\rho_k^3 = -1$. Note that

$$\frac{\partial G}{\partial \nu}(\nu, \eta) = 4\nu^3 + 1,$$

hence, using that ρ_k is a cubic root of -1, we obtain

$$\frac{\partial G}{\partial \nu}(\nu_k, \eta) = \begin{cases} 1, & k = 1, \\ -3, & k \in \{2, 3, 4\}. \end{cases}$$

Once again, the Implicit Function Theorem says that the four roots are simple, which gives the form of the approximate solutions (5.13) and (5.14).

5.5 Asymptotic analysis of the roots and the transfer function

Having a better knowledge of the roots, we will establish an asymptotic formula for the transfer function valid not only for s in a fixed compact subset of $\mathbb{C}$, but along a specific growing family of frequencies relevant to the stability analysis. This is mainly motivated by the order of the operator involved in the calculation as we will see soon.

Lemma 5.9 (Vieta identity)**.** *Let $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ be the four roots of (5.4). Then, exactly,*

$$\lambda_3 + \lambda_4 = -(\lambda_1 + \lambda_2), \qquad \text{and} \qquad \lambda_3^2 + \lambda_3\lambda_4 + \lambda_4^2 = \lambda_1\lambda_2. \tag{5.18}$$

Proof. Dividing (5.4) by η, the monic quartic $\lambda^4 + \frac{1}{\eta}\lambda + \frac{s}{\eta}$ has no cubic term and no quadratic term, so its elementary symmetric functions of the roots satisfy $e_1 := \sum_k \lambda_k = 0$ and $e_2 := \sum_{i<j} \lambda_i\lambda_j = 0$. The first identity in (5.18) is exactly $e_1 = 0$, rewritten. For the second, expand $e_2 = 0$ as

$$0 = \lambda_1\lambda_2 + (\lambda_1 + \lambda_2)(\lambda_3 + \lambda_4) + \lambda_3\lambda_4 = \lambda_1\lambda_2 - (\lambda_1 + \lambda_2)^2 + \lambda_3\lambda_4,$$

using $\lambda_3 + \lambda_4 = -(\lambda_1 + \lambda_2)$. Hence $\lambda_3\lambda_4 = (\lambda_1 + \lambda_2)^2 - \lambda_1\lambda_2$, and therefore

$$\lambda_3^2 + \lambda_3\lambda_4 + \lambda_4^2 = (\lambda_3 + \lambda_4)^2 - \lambda_3\lambda_4 = (\lambda_1 + \lambda_2)^2 - \left[(\lambda_1 + \lambda_2)^2 - \lambda_1\lambda_2\right] = \lambda_1\lambda_2.$$

□

Proposition 5.10. *Let s range over a fixed compact subset of $\mathbb{C}$. Then, as $\eta \to 0^+$,*

$$f_\eta(s) = e^{\lambda_1(s)}\left(1 + O(\eta^{2/3})\right), \tag{5.19}$$

where $\lambda_1(s)$ is the finite branch from Proposition 5.6. In particular, $f_\eta(s) \to e^{-s}$ uniformly on compacts.

Proof. Write the general solution of (5.2), with $y_i(s, 0) = 1$, as

$$y_i(x) = A_1\, e^{\lambda_1 x} + A_2\, e^{\lambda_2 x} + B_3\, e^{\lambda_3(x-1)} + B_4\, e^{\lambda_4(x-1)}, \tag{5.20}$$

where the layer terms at $x = 1$ have been rescaled by $e^{\lambda_{3,4}}$ so that their coefficients B_3, B_4 are $O(1)$ near $x = 1$. By Proposition 5.6, λ_1 is $O(1)$. Indeed, since $s \in K \subset \mathbb{C}$ where K is compact, we have $|s| \le R$ for $R > 0$. Hence by (5.13) and using $\eta \le 1$, we have

$$|\lambda_1(s)| \le |s| + M\eta \le K + M = O(1).$$

For the exploding branches we have

$$|e^{\lambda_2}| \le e^{-c\eta^{-1/3}}, \qquad |e^{-\lambda_3}|, |e^{-\lambda_4}| \le e^{-c\eta^{-1/3}} \tag{5.21}$$

for some $c > 0$ and η small from $\Re(\lambda_2) \to -\infty$, $\Re(\lambda_3), \Re(\lambda_4) \to +\infty$, see Remark 5.7. Indeed, for $\lambda_2(s)$ note that $\Re(\lambda_2) = -\eta^{-1/3} + R$ again by compactness of K. Is η is small enough, we can choose $\eta^{-1/3} \geq 2R$ which yields $\Re(\lambda_2) \leq -\frac{1}{2}\eta^{-1/3}$.

For $\lambda_{3,4}$ we have $\Re(\lambda_{3,4}) = \frac{1}{2}\eta^{-1/3} + R$, again by compactness of K. Taking η small enough to get $\eta^{-1/3} \geq 4R'$, we get

$$\Re(\lambda_{3,4}) \geq \frac{1}{2}\eta^{-1/3} - R' \geq \frac{1}{4}\eta^{-1/3}.$$

Applying the previous estimates on $\left|e^{\lambda_2}\right|$ and $\left|e^{\lambda_{3,4}}\right|$ yields the claimed estimates.

Now we attack the block of equations at the $x = 1$ layer. Imposing $\partial_x^2 y_i(1) = \partial_x^3 y_i(1) = 0$ gives the 2×2 system

$$\lambda_1^2 A_1 e^{\lambda_1} + \lambda_2^2 A_2 e^{\lambda_2} + \lambda_3^2 B_3 + \lambda_4^2 B_4 = 0, \qquad \lambda_1^3 A_1 e^{\lambda_1} + \lambda_2^3 A_2 e^{\lambda_2} + \lambda_3^3 B_3 + \lambda_4^3 B_4 = 0.$$

Writing this as

$$\begin{pmatrix} \lambda_3^2 & \lambda_4^2 \\ \lambda_3^3 & \lambda_4^3 \end{pmatrix} \begin{pmatrix} B_3 \\ B_4 \end{pmatrix} = -\sum_{i=1}^{2} \begin{pmatrix} \lambda_i^2 \\ \lambda_i^3 \end{pmatrix} c_{\lambda_i},$$

with weights $c_{\lambda_1} = A_1 e^{\lambda_1}$, $c_{\lambda_2} = A_2 e^{\lambda_2}$, and inverting the 2×2 matrix explicitly since the determinant of the matrix is $\lambda_3^2 \lambda_4^2 (\lambda_4 - \lambda_3) \neq 0$ gives, exactly,

$$\begin{pmatrix} B_3 \\ B_4 \end{pmatrix} = -\frac{1}{(\lambda_3\lambda_4)^2(\lambda_4 - \lambda_3)} \begin{pmatrix} \lambda_4^3 & -\lambda_4^2 \\ -\lambda_3^3 & \lambda_3^2 \end{pmatrix} \begin{pmatrix} \lambda_1^2 c_1 + \lambda_2^2 c_2 \\ \lambda_1^3 c_1 + \lambda_2^3 c_2 \end{pmatrix},$$

which yields

$$B_3 = \frac{c_1\lambda_1^2(\lambda_4 - \lambda_1) + c_2\lambda_2^2(\lambda_4 - \lambda_2)}{\lambda_3^2(\lambda_3 - \lambda_4)}, \quad B_4 = \frac{-c_1\lambda_1^2(\lambda_3 - \lambda_1) - c_2\lambda_2^2(\lambda_3 - \lambda_2)}{\lambda_4^2(\lambda_3 - \lambda_4)}. \tag{5.22}$$

Adding B_3 and B_4 gives

$$B_3 + B_4 = -\sum_{i=1}^{2} c_{\lambda_i} \frac{\lambda_i^2\left[\lambda_4^2(\lambda_4 - \lambda_i) + \lambda_3^2(\lambda_i - \lambda_3)\right]}{(\lambda_3\lambda_4)^2(\lambda_4 - \lambda_3)}. \tag{5.23}$$

The bracket in (5.23) would appear to be $O(\eta^{-1})$ since $\lambda_3, \lambda_4 \sim \eta^{-1/3}$, suggesting an $O(\eta^{1/3})$ contribution. This is where Lemma 5.9 intervenes. Factor the bracket algebraically, treating λ_i as an indeterminate for $i \in \{1, 2\}$,

$$\lambda_4^2(\lambda_4 - \lambda_i) + \lambda_3^2(\lambda_i - \lambda_3) = (\lambda_4^3 - \lambda_3^3) - \lambda_i(\lambda_4^2 - \lambda_3^2) = (\lambda_4 - \lambda_3)\Big[(\lambda_3^2 + \lambda_3\lambda_4 + \lambda_4^2) - \lambda_i(\lambda_3 + \lambda_4)\Big], \tag{5.24}$$

using $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$ and $a^2 - b^2 = (a - b)(a + b)$. Substituting the two exact identities (5.18) into the bracket,

$$(\lambda_3^2 + \lambda_3\lambda_4 + \lambda_4^2) - \lambda_i(\lambda_3 + \lambda_4) = \lambda_1\lambda_2 + \lambda_i(\lambda_1 + \lambda_2),$$

which involves only the two $O(1)$ and $O(\eta^{-1/3})$ terms λ_1, λ_2. The large quantities λ_3, λ_4 with $O(\eta^{-1/3})$ have disappeared entirely. Evaluating at $i = 1$ we obtain $\lambda_1(\lambda_1 + 2\lambda_2)$ and

at $i = 2$ the bracket becomes $\lambda_2(\lambda_2 + 2\lambda_1)$. Substituting back into (5.24) and (5.23), we obtain

$$B_3 + B_4 = -A_1 e^{\lambda_1} X_1 - A_2 e^{\lambda_2} X_2, \qquad X_1 := \frac{\lambda_1^3(\lambda_1 + 2\lambda_2)}{(\lambda_3\lambda_4)^2}, \quad X_2 := \frac{\lambda_2^3(\lambda_2 + 2\lambda_1)}{(\lambda_3\lambda_4)^2}. \tag{5.25}$$

Since $|\lambda_1| = O(1)$ on the fixed compact set, $|\lambda_2| \sim \eta^{-1/3}$, and $|(\lambda_3\lambda_4)^2| \sim \eta^{-4/3}$ (Proposition 5.6), we read off directly

$$X_1 = O(\eta), \qquad X_2 = O(\eta). \tag{5.26}$$

Now we focus on the system generated at the $x = 0$ layer.

Imposing $y_i(0) = 1$ and $\partial_x^2 y_i(0) = 0$ we have

$$\begin{aligned} \lambda_1^2 A_1 + \lambda_2^2 A_2 + \lambda_3^2 B_3 e^{-\lambda_3} + \lambda_4^2 B_4 e^{-\lambda_4} &= 0, \\ A_1 + A_2 + B_3 e^{-\lambda_3} + B_4 e^{-\lambda_4} &= 1. \end{aligned}$$

We claim that substituting (5.25) for $B_3 + B_4$ gives, into the first equation,

$$A_1\lambda_1^2 + A_2\lambda_2^2 = 0 + O\left(e^{-c\eta^{-1/3}}\right) \implies A_2 = -A_1\frac{\lambda_1^2}{\lambda_2^2} + O\left(e^{-c\eta^{-1/3}}\right),$$

and in the second equation

$$A_1 + A_2 = 1 + O\left(e^{-c\eta^{-1/3}}\right),$$

where the last parenthesis is exponentially small since each term carries $e^{-\lambda_3}$ or $e^{-\lambda_4}$ and is discarded. Indeed, first note that by (5.22) and (5.21),

$$B_3 e^{-\lambda_3} = A_1 e^{\lambda_1 - \lambda_3} X_1 + A_2 e^{\lambda_2 - \lambda_3} X_2 = o(1).$$

The same argument shows $B_4 e^{-\lambda_4} = o(1)$, which gives

$$A_1 + A_2 = 1 + O\left(e^{-c\eta^{-1/3}}\right), \qquad \lambda_1^2 A_1 + \lambda_2^2 A_2 = O\left(e^{-c\eta^{-1/3}}\right).$$

Hence, substituting A_2,

$$A_1\Big(1 - \frac{\lambda_1^2}{\lambda_2^2}\Big) = 1 + O(e^{-c\eta^{-1/3}}) \implies A_1 = \frac{1}{1 - (\lambda_1/\lambda_2)^2}(1 + O(e^{-c\eta^{-1/3}})).$$

Using $|\lambda_1/\lambda_2| = O(\eta^{1/3})$ on the compact set, since $\lambda_1(s) = -s + O(\eta)$ and $\lambda_2(s) = -\eta^{-1/3} + \frac{s}{3} + O(\eta^{2/3})$,

$$A_1 = 1 + \Big(\frac{\lambda_1}{\lambda_2}\Big)^2 + O(\eta^{2/3}) = 1 + O(\eta^{2/3}), \qquad A_2 = -\Big(\frac{\lambda_1}{\lambda_2}\Big)^2 A_1 = O(\eta^{2/3}). \tag{5.27}$$

Now we approximate the transfer function f_η knowing the approximations given by $y_i(s, 1)$ and $y_i(s, 0) = 1$.

$$f_\eta(s) = y_i(1) = A_1 e^{\lambda_1} + A_2 e^{\lambda_2} + B_3 + B_4.$$

The middle term is $O(\eta^{2/3}) \cdot O(e^{-c\eta^{-1/3}})$, which is exponentially small, hence negligible next to the other two. Substituting (5.25) and (5.27):

$$\begin{aligned} f_\eta(s) &= A_1 e^{\lambda_1}(1 - X_1) + A_2 e^{\lambda_2}(1 - X_2) \\ &= e^{\lambda_1}(1 + s^2\eta^{2/3} + O(\eta))(1 - O(\eta)) + o(1) \\ &= e^{\lambda_1}(1 + O(\eta^{2/3})). \end{aligned}$$

using (5.26) for $X_1 = O(\eta)$, which is strictly smaller than the $O(\eta^{2/3})$ correction from A_1 and is absorbed into it since we are taking $\eta \to 0$. This proves (5.19). Finally, $\lambda_1(s) = -s + O(\eta)$ from Proposition 5.6 gives $e^{\lambda_1(s)} = e^{-s}(1 + O(\eta))$, and combining with (5.19), we obtain that $f_\eta(s) \to e^{-s}$ uniformly on the compact set, at rate $O(\eta^{2/3})$. □

We now extend Proposition 5.10 not only for s in a compact subset, but along the specific growing family of frequencies relevant to the stability analysis.

Proposition 5.11. *Let $\psi > 0$ be fixed and set*

$$s = s_\eta(\sigma, \psi) := \sigma + i\eta^{-1/4}\psi, \qquad \sigma \in \mathbb{R} \text{ fixed.}$$

Then, as $\eta \to 0^+$,

$$\eta s^4 \to \psi^4, \tag{5.28}$$

and

$$f_\eta(s) = e^{-s-\psi^4+o(1)}, \qquad \eta \to 0^+. \tag{5.29}$$

Proof. Expand $s^4 = (\sigma + i\eta^{-1/4}\psi)^4$ by the binomial theorem,

$$s^4 = \eta^{-1}\psi^4 - 4i\sigma\,\eta^{-3/4}\psi^3 - 6\sigma^2\eta^{-1/2}\psi^2 + 4i\sigma^3\eta^{-1/4}\psi + \sigma^4,$$

using $i^4 = 1$. Multiplying by η and taking the limit $\eta \to 0^+$,

$$\eta s^4 = \psi^4 - 4i\sigma\,\eta^{1/4}\psi^3 - 6\sigma^2\eta^{1/2}\psi^2 + 4i\sigma^3\eta^{3/4}\psi + \sigma^4\eta \;\longrightarrow\; \psi^4,$$

since every term but the first carries a positive power of η. This proves (5.28).

First we see that the expansion of the finite branch is valid at this scale. The remainder term in (5.15) is $O(\eta^2 s^7)$. With $|s| \sim \eta^{-1/4}$, $\eta^2 s^7 = O(\eta^{2-7/4}) = O(\eta^{1/4}) \to 0$, so (5.15) remains valid and

$$\lambda_1(s) = -s - \eta s^4 + o(1) = -s - \psi^4 + o(1).$$

Now we see the exploding branch case. The remainder in (5.16) requires $s\eta^{1/3} \to 0$, here $s\eta^{1/3} = O(\eta^{-1/4+1/3}) = O(\eta^{1/12}) \to 0$. Hence $\lambda_k(s)$, $k = 2, 3, 4$, retains the same qualitative behavior since real parts take values $\mp\infty$ according to the signs of $\Re(\rho_k)$ for $k = 2, 3, 4$.

Finally, we need to study the negligibility of the layer corrections at this scale. The $x = 0$ correction found in the proof of Proposition 5.10 above was $O((\lambda_1/\lambda_2)^2)$, with $|\lambda_1| \sim |s| \sim \eta^{-1/4}$ and $|\lambda_2| \sim \eta^{-1/3}$, this is $O(\eta^{-1/2} \cdot \eta^{2/3}) = O(\eta^{1/6}) \to 0$. The $x = 1$ correction X_1 is $O(\eta^{1/4})$ at this scale, remaining strictly smaller than the $O(\eta^{1/6})$ term. Hence both layer corrections vanish at this scale, the $x = 0$ correction at rate $\eta^{1/6} \to 0$ being the dominant one, and consequently

$$f_\eta(s) = e^{\lambda_1(s)}(1 + o(1)) = e^{-s-\psi^4+o(1)},$$

which is (5.29). □

5.6 Main result and relevant lemmas

In this section we prove Theorem 5.5. To do this, we first prove some structural lemmas similar to the strategy followed in Chapter 3 and Chapter 4.

Let us first consider a sequence

$$(\eta_n)_{n\in\mathbb{N}} : \ \eta_n > 0, \ \forall n \in \mathbb{N} \text{ and } \lim_{n\to+\infty} \eta_n = 0^+, \tag{5.30}$$

and an associated sequence of system poles

$$(s_n)_{n\in\mathbb{N}} : \quad s_n \in \mathcal{S}_{\eta_n}, \ \forall n \in \mathbb{N}. \tag{5.31}$$

In order to prove Theorem 5.5, we will look at the adherent points of the sequences $(s_n)_{n\in\mathbb{N}}$ when $n \to +\infty$, which implies $\eta_n \to 0^+$. By definition, we know that $\overline{s}$ is an adherent point of a sequence $(s_n)_{n\in\mathbb{N}}$ if and only if there exists a subsequence which converges to $\overline{s}$. With a slight abuse of notation, we will write $s_n \to \overline{s}$ to signify that $\overline{s}$ is an adherent point of the sequence $(s_n)_{n\in\mathbb{N}}$, but it is implied that the convergence in fact only relies on the adequate subsequence.

Now fix a sequence $\eta_n \to 0^+$ and an associated sequence $s_n \in \mathcal{S}_{\eta_n}$. Write $s_n = \sigma_n + i\omega_n$.

Lemma 5.12. *Let $(\eta_n)_{n\in\mathbb{N}}$ be a sequence of the form (5.30) and $(s_n)_{n\in\mathbb{N}}$ be an associated sequence of the form (5.31). Then*

1. *$|s_n| \to +\infty$. In other words, $(s_n)_{n\in\mathbb{N}}$ has no bounded subsequence.*
2. *If $(\sigma_n)_{n\in\mathbb{N}}$ has an accumulation point, then $|\omega_n| \to +\infty$ along that subsequence.*

Proof. If (s_{n_k}) is a bounded subsequence with $s_{n_k} \to \overline{s} \in \mathbb{C}$, then by Proposition 5.10, $f_{\eta_{n_k}}(s_{n_k}) \to e^{-\overline{s}} = f_0(\overline{s})$. Passing to the limit in the characteristic equation (5.12),

$$0 = f_0^2(\overline{s}) - f_0(\overline{s})e^{-\overline{s}} - 1 = e^{-2\overline{s}} - e^{-2\overline{s}} - 1 = -1,$$

which is a contradiction. Hence no bounded subsequence of (s_n) can converge to a finite point. Since $\eta_n \to 0^+$ forces $\Re(s_n)$ to remain bounded above along any convergent subsequence, the only possibility for a bounded subsequence is $\Re(s_n) \to -\infty$. This proves the first statement.

The second is an immediate consequence, since ω_n bounded together with $\Re(s_n)$ bounded would make (s_n) itself bounded. Indeed, let $\overline{s} \in \mathbb{C}$ such that $s_{n_k} \to \overline{s}$. We know that

$$|s_{n_k}|^2 = s_{n_k} \cdot \overline{s_{n_k}} = \Re(s_{n_k})^2 + \Im(s_{n_k})^2 = \sigma_{n_k}^2 + \omega_{n_k}^2.$$

By contradiction suppose that $\omega_{n_k} \to \overline{\omega} \in \mathbb{R}$. Then it is bounded, as well as σ_{n_k}. Therefore s_{n_k} is bounded and by the first point, $\Re(s_{n_k}) \to -\infty$, which is a contradiction. This proves $\omega_{n_k} \to +\infty$. □

Lemma 5.13. *Let $(\eta_n)_{n\in\mathbb{N}}$ be a sequence of the form (5.30) and $(s_n)_{n\in\mathbb{N}}$ be an associated sequence of the form (5.31). Denote $\sigma_n = \Re(s_n)$ and $\omega_n = \Im(s_n)$. Suppose $\omega_n \to +\infty$ and set*

$$t_n := \eta_n \omega_n^4.$$

Let $\bar{t} \in [0, +\infty]$ be an accumulation point of $(t_n)_{n\in\mathbb{N}}$, along a subsequence such that $s_n \to \bar{s} \in \mathbb{C} \cup \{\infty\}$ appropriately. Then

1. *If $\bar{t} = 0$ or $\bar{t} = +\infty$, then $\sigma_n \to -\infty$ along the subsequence.*
2. *If $\bar{t} \in (0, +\infty)$, then*
$$\sigma_n \to g(\bar{t}) := -\frac{\bar{t}}{2} + \frac{1}{2}\ln(1 - e^{-\bar{t}}).$$

Proof. By Lemma 5.12 we may assume $\omega_n \to \infty$. Write $\omega_n = \eta_n^{-1/4}\psi_n$, so that $t_n = \psi_n^4$ by definition, and $\psi_n \to \bar{t}^{1/4}$ along the relevant subsequence. We separate the proof in three cases.

Case $\bar{t} = 0$. Then $\psi_n \to 0$, which gives $\omega_n = o(\eta_n^{-1/4})$. We claim $\Re(s_n) \to -\infty$. Indeed, suppose towards a contradiction that $\Re(s_n) = \sigma_n$ has a finite accumulation point σ^* along a further subsequence. By Proposition 5.11 applied with $\psi = \psi_n \to 0$, we get $\eta_n s_n^4 \to 0$, hence
$$f_{\eta_n}(s_n) = e^{-s_n - o(1)}.$$
Substituting into the characteristic equation (5.12) and taking the limit along the subsequence where $\sigma_n \to \sigma^*$, $\psi_n \to 0$,
$$e^{-2s_n+2o(1)} - e^{-2s_n+o(1)} - 1 = e^{-2s_n}(e^{2(o(1))} - e^{o(1)}) - 1 = 0.$$
Since $|e^{-2s_n}| = e^{-2\sigma_n} \to e^{-2\sigma^*}$ is bounded and $e^{2(o(1))} - e^{o(1)} \to 0$, we get $-1 = 0$, which is a contradiction.

Case $\bar{t} = +\infty$. Then $\psi_n \to +\infty$. By Proposition 5.11,
$$f_{\eta_n}(s_n) = e^{-s_n - \psi_n^4 + o(1)}.$$
Substituting into (5.12) and taking the modulus,
$$e^{-2\sigma_n}e^{-\psi_n^4}(1 - e^{-\psi_n^4}) = 1 + o(1).$$
As $\psi_n \to \infty$, $e^{-\psi_n^4}(1 - e^{-\psi_n^4}) \to 0$, forcing $e^{-2\sigma_n} \to +\infty$, which yields $\Re(s_n) = \sigma_n \to -\infty$.

Case $\bar{t} \in (0, \infty)$. By Proposition 5.11, $f_{\eta_n}(s_n) = e^{-s_n - \bar{t} + o(1)}$ where we are using $\psi_n^4 \to \bar{t}$. Substituting into (5.12):
$$e^{-2s_n}e^{-2\bar{t}}(1 + o(1)) - e^{-2s_n}e^{-\bar{t}}(1 + o(1)) - 1 = o(1),$$
rearranging,
$$e^{-2s_n}\left[e^{-2\bar{t}} - e^{-\bar{t}}\right] = 1 + o(1). \tag{5.32}$$
Writing $s_n = \sigma_n + i\omega_n$ and taking moduli in (5.32) yields
$$e^{-2\sigma_n}\left|e^{-2\bar{t}} - e^{-\bar{t}}\right| = e^{-2\sigma_n}\, e^{-\bar{t}}(1 - e^{-\bar{t}}) = 1 + o(1)$$
Solving for σ_n,
$$-2\sigma_n = \bar{t} - \ln\left(1 - e^{-\bar{t}}\right) + o(1) \implies \sigma_n \longrightarrow -\frac{\bar{t}}{2} + \frac{1}{2}\ln\left(1 - e^{-\bar{t}}\right) = g(\bar{t}),$$
which is the claimed limit. □

Lemma 5.14. *Let $(\eta_n)_{n\in\mathbb{N}}$ be a sequence of the form (5.30) and $(s_n)_{n\in\mathbb{N}}$ be an associated sequence of the form (5.31). Denote $\sigma_n = \Re(s_n)$ and $\omega_n = \Im(s_n)$. Then every accumulation point $\overline{s}$ of $(s_n)_{n\in\mathbb{N}}$ satisfies*

$$\Re(\overline{s}) \leq -\ln(2).$$

Proof. Let $\overline{s}$ be an accumulation point of (s_n), attained along a subsequence with $s_n \to \overline{s}$ allowing $\Re(\overline{s}) = -\infty$. By Lemma 5.12, $\omega_n \to \infty$ along this subsequence or $\Re(s_n) \to -\infty$ directly, in which case $\Re(\overline{s}) = -\infty \leq -\ln(2)$ directly. Assume then $\omega_n \to \infty$, and pass to a further subsequence along which $t_n = \eta_n \omega_n^4$ converges to some $\overline{t} \in [0, +\infty]$. By Lemma 5.13,

- If $\overline{t} \in \{0, +\infty\}$: $\Re(s_n) \to -\infty$, so $\Re(\overline{s}) = -\infty \leq -\ln(2)$.
- If $\overline{t} \in (0, \infty)$: $\Re(s_n) \to g(\overline{t})$, so $\Re(\overline{s}) = g(\overline{t})$.

It remains to bound $g(\overline{t})$ for $\overline{t} \in (0, \infty)$. Differentiating,

$$g'(t) = -\frac{1}{2} + \frac{e^{-t}}{2(1-e^{-t})} = -\frac{1}{2} + \frac{1}{2(e^t - 1)},$$

which is strictly decreasing in t. Going one step further we obtain

$$g''(t) = -\frac{e^t}{2(e^t-1)},$$

which is strictly negative for $t > 0$. This means that $g(t)$ is concave and strictly decreasing, so there only one critical point t^* which reaches the maximum of g. This t^* solves $g'(t^*) = 0$, which gives

$$e^{t^*} - 1 = 1,$$

this yields $t^* = \ln(2)$. Evaluating,

$$g(\ln(2)) = -\frac{\ln(2)}{2} + \frac{1}{2}\ln\left(1 - \frac{1}{2}\right) = -\frac{\ln(2)}{2} - \frac{\ln(2)}{2} = -\ln(2).$$

Therefore $g(\overline{t}) \leq g(\ln(2)) = -\ln(2)$ for every $\overline{t} \in (0, \infty)$, so in every case $\Re(\overline{s}) \leq -\ln(2)$. □

Now we are in position to prove the main result of the chapter.

Proof of Theorem 5.5. Suppose, towards a contradiction, that the conclusion fails for some $\delta > 0$. This means that for every $\eta_1 > 0$ there exist $\eta \in (0, \eta_1)$ and $s \in \mathcal{S}_\eta$ with $\Re(s) > -\ln(2) + \delta$. Repeating this process with $\eta_n = 1/n$ for each $n \in \mathbb{N}$, we construct a sequence $\eta_n \to 0^+$ and $s_n \in \mathcal{S}_{\eta_n}$ with

$$\Re(s_n) > -\ln(2) + \delta \qquad \forall n \in \mathbb{N}.$$

Hence (s_n) admits an accumulation point $\overline{s}$ with $\Re(\overline{s}) \geq -\ln(2) + \delta$. This contradicts Lemma 5.14. Hence the theorem holds. □

5.7 Robustness analysis

We consider again the open-loop system (5.1), but since we want to explicitly account for the sensitivity to delay uncertainties, we introduce an additional perturbation ε to the nominal transport velocity $v = 1$,

$$\begin{cases} \partial_t y_1(t,x) + (1+\varepsilon)\partial_x y_1(t,x) + \eta \partial_x^4 y_1(t,x) = 0, & t > 0,\ x \in (0,1), \\ \partial_t y_2(t,x) + (1+\varepsilon)\partial_x y_2(t,x) + \eta \partial_x^4 y_2(t,x) = 0, & t > 0,\ x \in (0,1), \\ y_1(t,0) = y_2(t,1) + U(t), \\ y_2(t,0) = y_1(t,1), \\ \partial_x^2 y_i(t,0) = \partial_x^2 y_i(t,1) = \partial_x^3 y_i(t,1) = 0, \quad i = 1,2, \\ Y(t) = y_1(t,1). \end{cases} \tag{5.33}$$

As before, we close the system with the dead beat output feedback controller $U(t) = -Y(t-1)$. This controller depends on the theoretical delay $\tau = 1$ ignoring the uncertainty represented by ε.

We will also assume that we have the same uncertainty ε for both physical subsystems represented by the transport equations with fourth-order perturbation. The perturbed transfer function $f_{\eta,\varepsilon}(s)$ is defined in the same way as in (5.5), but with $\lambda_k(s)$ solving

$$\eta\lambda^4 + (1+\varepsilon)\lambda + s = 0, \tag{5.34}$$

and the closed-loop characteristic equation retains the same form,

$$f_{\eta,\varepsilon}(s)^2 - f_{\eta,\varepsilon}(s)e^{-s} - 1 = 0. \tag{5.35}$$

Now we introduce the spectrum $\mathcal{S}_{\eta,\varepsilon}$ and maximal spectral abscissa $\sigma_{\eta,\varepsilon}$, defined by

$$\mathcal{S}_{\eta,\varepsilon} = \{s \in \mathbb{C} :\ s \text{ is solution of } (5.35)\}, \tag{5.36}$$

$$\sigma_{\eta,\varepsilon} = \sup\{\Re(s) :\ s \in \mathcal{S}_{\eta,\varepsilon}\}. \tag{5.37}$$

We remark that, by definition, we have

$$\mathcal{S}_{\eta,0} = \mathcal{S}_\eta, \quad \text{and} \quad \sigma_{\eta,0} = \sigma_\eta. \tag{5.38}$$

Also, in both cases $\varepsilon > 0$ and $\varepsilon = 0$ we do not have a new boundary-layer effect. In particular, by compactness we know that $f_{\eta,\varepsilon}(s) \to f_{\eta,0}(s) = f_\eta$ as $\varepsilon \to 0^+$. This is important because we can still make use of the approximation results coming from Section 5.4 and Section 5.5.

We then have the following robustness theorem.

Theorem 5.15 (Robustness)**.** *Let $\delta > 0$ and $\eta > 0$ such that $\sigma_{\eta,0} \le -\ln(2) + \delta$. Then there exists $\varepsilon_1 > 0$ such that*

$$\sigma_{\eta,\varepsilon} \le -\ln(2) + 2\delta, \qquad \forall \varepsilon \in (-\varepsilon_1, \varepsilon_1). \tag{5.39}$$

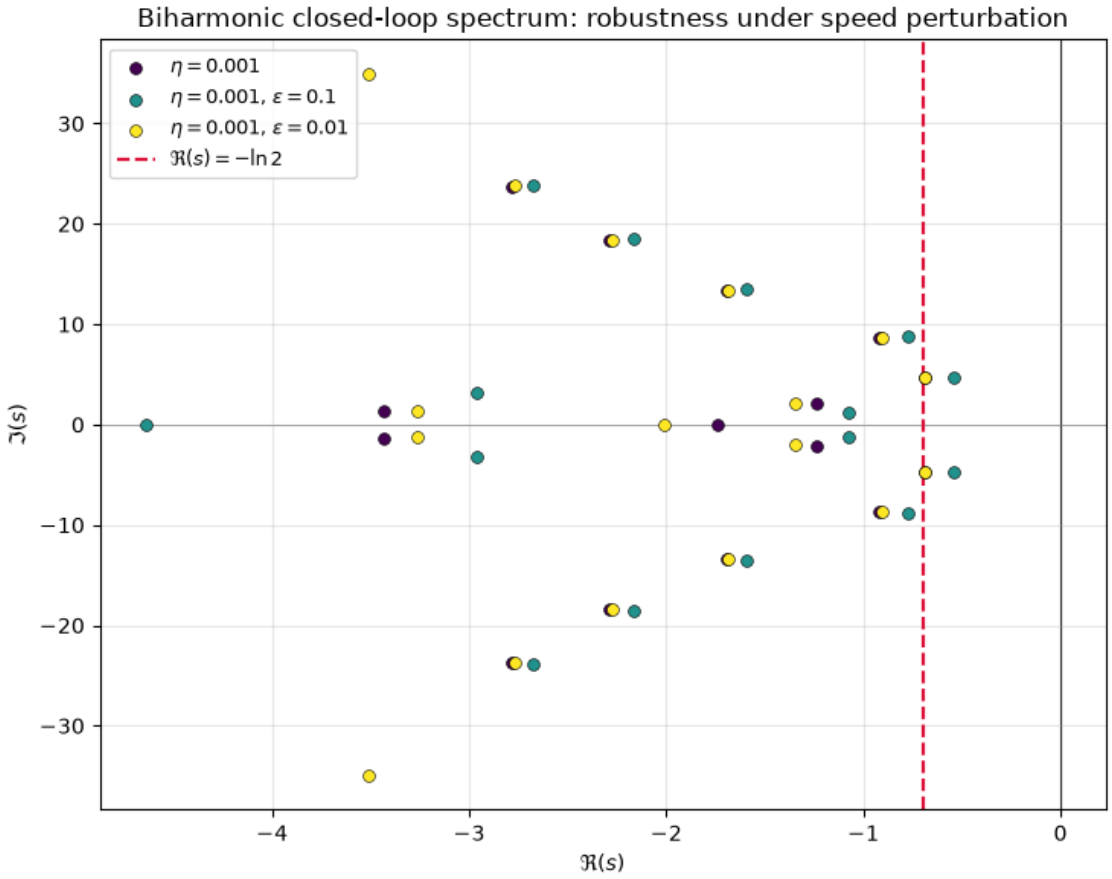


Figure 5.3: Perturbed closed-loop spectrum of (5.33)

Proof. Suppose, towards a contradiction, that no such ε_1 exists. Then there is a sequence $\varepsilon_n \to 0$ and $s_n \in \mathcal{S}_{\eta,\varepsilon_n}$ with

$$\Re(s_n) > -\ln(2) + 2\delta \qquad \forall n \in \mathbb{N}. \tag{5.40}$$

Case 1: (s_n) bounded. Extract $s_n \to \overline{s} \in \mathbb{C}$. Since $f_{\eta,\varepsilon_n}(s_n) \to f_\eta(\overline{s})$ uniformly, we pass to the limit in (5.35), $\overline{s} \in \mathcal{S}_{\eta,0} = \mathcal{S}_\eta$, and by the hypothesis on η, δ, $\Re(\overline{s}) \leq -\ln(2) + \delta$. But passing to the limit in (5.40) gives $\Re(\overline{s}) \geq -\ln(2) + 2\delta > -\ln(2) + \delta$, which yields a contradiction.

Case 2: (s_n) unbounded. Extract $r_n := |s_n| \to \infty$.

Write $s_n = r_n e^{i\theta_n}$, $\theta_n \in (-\pi, \pi]$. By (5.40), $r_n \cos\theta_n = \Re(s_n) > -\ln(2) + 2\delta$, so

$$\cos\theta_n > \frac{-\ln(2) + 2\delta}{r_n} \xrightarrow[n\to\infty]{} 0.$$

Hence, for n large enough that $\frac{\ln(2)}{r_n} < \frac{1}{2}$, $\cos\theta_n > -\frac{1}{2}$, i.e.,

$$\theta_n \in \Big(-\frac{2\pi}{3}, \frac{2\pi}{3}\Big). \tag{5.41}$$

With η fixed, set $N_n := (r_n/\eta)^{1/4} \to \infty$ and $s_n/r_n = e^{i\theta_n}$ in order to rescale to the quartic.

Substituting $\lambda = N_n\xi$ into $\eta\lambda^4 + (1+\varepsilon_n)\lambda + s_n = 0$ and dividing by $r_n = \eta N_n^4$:

$$\xi^4 + e^{i\theta_n} + \frac{(1+\varepsilon_n)N_n}{r_n}\,\xi = 0, \qquad \frac{(1+\varepsilon_n)N_n}{r_n} = \frac{1+\varepsilon_n}{\eta^{1/4} r_n^{3/4}} \to 0. \tag{5.42}$$

By (5.41), along any subsequence $e^{i\theta_n} \to e^{i\alpha}$ with $\alpha \in [-2\pi/3, 2\pi/3]$ bounded away from $\alpha = \pi$ by a fixed margin $\pi/3$. Avoiding this direction is important since, when taking the limit, (5.42) becomes $\xi^4 = -e^{i\alpha}$. These four roots, which yields the four directions of the branches, are separated by $\pi/2$ along the complex plane. Denoting by ξ_j for $j \in \{0,1,2,3\}$ the four roots of $-e^{i\alpha}$, we note that some ξ_j becomes purely imaginary if and only if $\alpha \equiv \pi \mod 2\pi$. Indeed, since

$$\xi_j = \exp\left[i\frac{\pi + \alpha + 2\pi j}{4}\right], \quad j = 0,1,2,3,$$

ξ_j is purely imaginary if and only if $\frac{\pi+\alpha+2\pi j}{4} = \frac{\pi}{2} + m\pi$ for some $m \in \mathbb{Z}$, which yields $\alpha \equiv \pi \mod 2\pi$.

Now, since $\alpha \in [-2\pi/3, 2\pi/3]$ and the separation of the roots ξ_j by $\pi/2$ radians between the four directions, the split of the four limiting roots $\xi_j = e^{i(\pi+\alpha+2\pi j)/4}$ of $\xi^4 = -\hat{\tau}_0$. When we look at the four branches, two have $\Re(\lambda_j) \to -\infty$ and two have $\Re(\lambda_j) \to +\infty$. Following the proof of Proposition 5.10, this forces their amplitudes to vanish to leading order in (5.20), and hence

$$f_{\eta,\varepsilon_n}(s_n) \longrightarrow 0. \tag{5.43}$$

Now we need to show that e^{-s_n} is bounded. From (5.40), $\Re(s_n) > -\ln(2) + 2\delta$ for every $n \in \mathbb{N}$, so

$$|e^{-s_n}| = e^{-\Re(s_n)} < e^{\ln(2)-2\delta} \qquad \forall n \in \mathbb{N}, \tag{5.44}$$

which proves that e^{-s_n} is uniformly bounded.

Finally, substituting (5.43) and (5.44) into (5.35) we get that since e^{-s_n} is bounded and $f_{\eta,\varepsilon_n}(s_n) \to 0$,

$$0 = f_{\eta,\varepsilon_n}^2(s_n) - f_{\eta,\varepsilon_n}(s_n)\, e^{-s_n} - 1 \longrightarrow 0 - 0 - 1 = -1 \neq 0,$$

which gives a contradiction. Since both cases yield a contradiction, the Theorem is proved. □

5.8 Sharpness of the constant $-\ln(2)$

In Section 5.6 and Section 5.7 we proved the robustness of the stabilization for the system (5.1) under perturbations on the nominal velocity for the transport term, adding a fourth-order perturbation. In Chapter 3 a similar strategy is developed, proving upper bounds for the maximal spectral abscissa. A conjecture is provided, saying that in the second-order perturbation case $\ln(2)$ is the optimal constant by numerical experiments.

The previous conjecture is solved in Chapter 4, for a different system using the topological degree machinery. It is also mentioned in Remark 4.10, that one could use the same

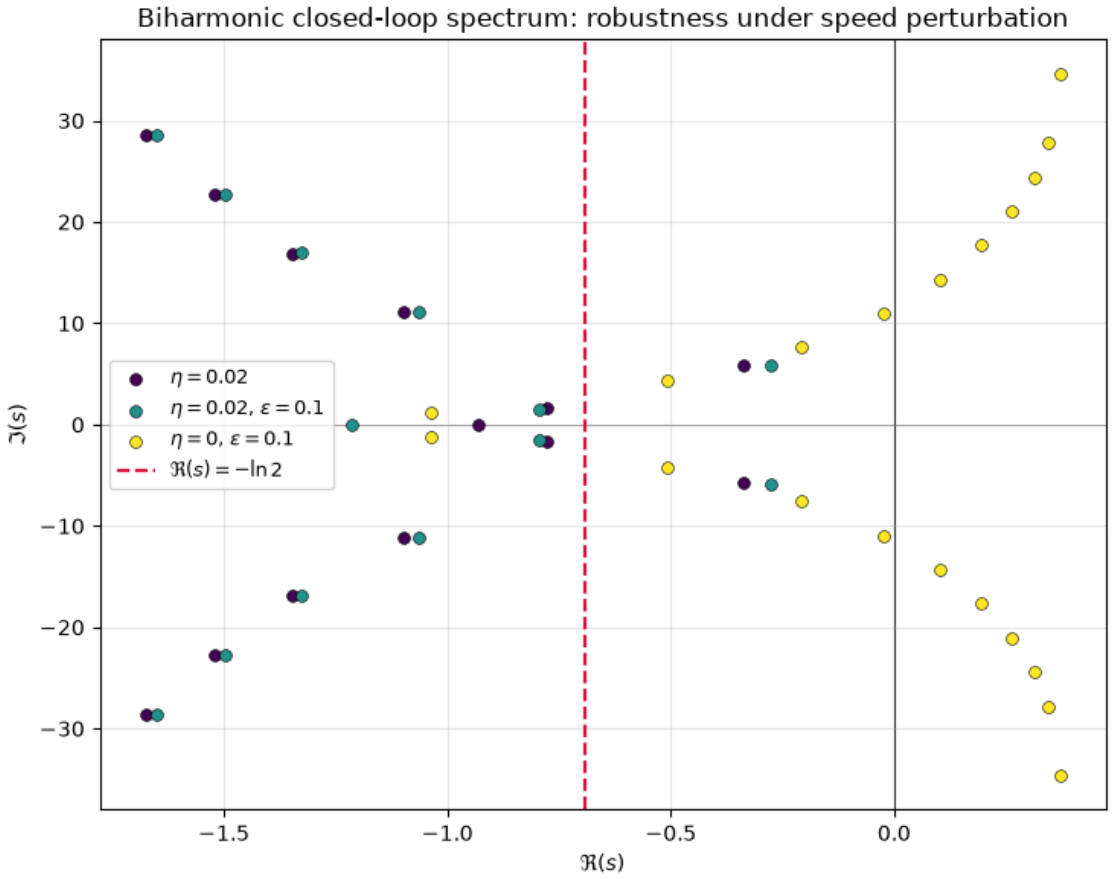


Figure 5.4: Perturbed closed-loop spectrum of (5.33)

strategy to solve Conjecture 1 from Chapter 3.

In this Chapter we prove the same conjecture, but for the fourth-order perturbation case. The surprise in this setting is that the same constant is the optimal, even knowing that the different perturbations differs in order. The theorem is the following.

Theorem 5.16 (Sharpness)**.** *For every $\delta > 0$, there exists $\eta_0 > 0$ such that for every $\eta \in (0, \eta_0)$, there exists $\tilde{s} \in \mathcal{S}_\eta$ with*

$$|\Re(\tilde{s}) + \ln(2)| < \delta. \tag{5.45}$$

Consequently, choosing $\tilde{\eta} = \min\{\eta_0, \eta_1\}$ where η_1 comes from Theorem 5.5, we have

$$-\ln(2) - \delta \leq \sigma_\eta \leq -\ln(2) + \delta, \qquad \forall \eta \in (0, \tilde{\eta}). \tag{5.46}$$

In other words, the root $\tilde{s} \in \mathcal{S}_\eta$ can be located in a resonant band: writing $\tilde{s} = \tilde{\sigma} + i\tilde{\omega}$ we have $\eta\tilde{\omega}^4 \to \ln(2)$ as $\eta \to 0^+$.

Proof. We separate the proof in several steps.
Step 1: rescaled variables and the critical point.

The resonance analysis of Lemma 5.13 identified the relevant scale $\omega \sim \eta^{-1/4}$. Accordingly, set

$$\gamma := \eta^{1/4}, \qquad \varphi := \gamma\omega, \qquad s = \sigma + i\,\frac{\varphi}{\gamma}.$$

This means $\varphi^4 = \eta\omega^4 = t$. The critical point identified in Lemma 5.13 given by $\varphi_*^4 = t^* = \ln(2)$, $\sigma^* = g(\ln(2)) = -\ln(2)$ becomes, in these variables,

$$(\sigma_*, \varphi_*) = \big(-\ln(2),\ (\ln(2))^{1/4}\big).$$

Step 2: factorization into a fast phase and a slow amplitude.

Define the normalized amplitude

$$h_\gamma(\sigma, \varphi) := e^{2s}\Big(f_\eta^2(s) - f_\eta(s)e^{-s}\Big)\Big|_{s=\sigma+i\varphi/\gamma,\ \eta=\gamma^4}.$$

By Proposition 5.11 $f_\eta(s) = e^{-s-\varphi^4}(1 + o(1))$, uniformly for (σ, φ) in compact subsets of $\mathbb{R} \times (0, \infty)$, which yields

$$f_\eta^2 = e^{-2s-2\varphi^4}(1 + o(1)), \qquad f_\eta e^{-s} = e^{-2s-\varphi^4}(1 + o(1)).$$

Near the critical point both terms are of the same order as their difference at $t^* = \ln(2)$ is $\frac{1}{4}$ since $e^{-2t^*} = \frac{1}{4}$, $e^{-t^*} = \frac{1}{2}$, so the relative errors survive the subtraction when we take $\gamma \to 0^+$,

$$h_\gamma(\sigma, \varphi) \ \longrightarrow\ h_0(\varphi) := e^{-2\varphi^4} - e^{-\varphi^4} \ <\ 0. \tag{5.47}$$

This last uniform convergence occurs on a compact set. Since $e^{-2s} = e^{-2\sigma}e^{-2i\varphi/\gamma}$, the characteristic function factors as

$$\mathcal{G}(s) := f_\eta(s)^2 - f_\eta(s)e^{-s} - 1 = e^{-2\sigma}e^{-2i\varphi/\gamma}\,h_\gamma(\sigma, \varphi) - 1 = -\Big(e^{-2i\varphi/\gamma}\,\mathcal{A}_\gamma(\sigma, \varphi) + \mathcal{B}\Big),$$

with

$$\mathcal{A}_\gamma(\sigma, \varphi) := -e^{-2\sigma}\,h_\gamma(\sigma, \varphi), \qquad \mathcal{B} := 1. \tag{5.48}$$

By (5.47), $\mathcal{A}_\gamma(\sigma, \varphi) \to \mathcal{A}_0(\sigma, \varphi) := e^{-2\sigma}\big(e^{-\varphi^4} - e^{-2\varphi^4}\big) > 0$ uniformly on compacts when $\gamma \to 0^+$. The zeros of $\mathcal{G}$ coincide with the zeros of $e^{-2i\varphi/\gamma}\mathcal{A}_\gamma + 1$, which takes the form of a rapidly rotating phase making one full turn as φ traverses an interval of length $\pi\gamma$, i.e., ω traverses an interval of length π, multiplying by a slowly varying amplitude.

Remark 5.17 (The identification $\Psi : \mathbb{C} \to \mathbb{R}^2$)**.** Since $e^{-2i\varphi/\gamma}\mathcal{A}_\gamma(\sigma, \varphi) + \mathcal{B}$ is complex-valued while the topological degree theory requires a map between real Euclidean spaces of equal dimension, we fix the canonical real isomorphism $\Psi(x + iy) := (x, y)$, $x, y \in \mathbb{R}$, and define $\mathcal{H}_\gamma : \Omega_\gamma \to \mathbb{R}^2$ by

$$\mathcal{H}_\gamma(\sigma, \varphi) := \Psi\Big(e^{-2i\varphi/\gamma}\mathcal{A}_\gamma(\sigma, \varphi) + \mathcal{B}\Big).$$

Writing $\mathcal{A}_\gamma = P_\gamma + iQ_\gamma$ with real functions $P_\gamma := \Re(\mathcal{A}_\gamma)$ and $Q_\gamma := \Im(\mathcal{A}_\gamma)$. Note that $P_\gamma(\sigma, \varphi) \to \mathcal{A}_0(\sigma, \varphi)$ and $Q_\gamma(\sigma, \varphi) \to 0$ by (5.47).

Expanding with $\theta := 2\varphi/\gamma$ gives

$$e^{-2i\varphi/\gamma}\mathcal{A}_\gamma(\sigma, \varphi) = (\cos\theta - i\sin\theta)(P_\gamma(\sigma, \varphi) + iQ_\gamma(\sigma, \varphi)),$$

the two real components are, explicitly,

$$\mathcal{H}_\gamma(\sigma, \varphi) = \Big(\cos\theta\,P_\gamma(\sigma, \varphi) + \sin\theta\,Q_\gamma(\sigma, \varphi) + 1,\quad \cos\theta\,Q_\gamma(\sigma, \varphi) - \sin\theta\,P_\gamma(\sigma, \varphi)\Big). \tag{5.49}$$

A zero of $\mathcal{H}_\gamma$ is, by construction of Ψ, exactly a zero of the original complex expression.

Step 3: verification of the two structural hypotheses from Remark 4.10

(H1) Amplitude matching at the critical point. With $t_* = \varphi_*^4 = \ln(2)$:

$$e^{-t_*} - e^{-2t_*} = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}, \qquad e^{-2\sigma_*} = e^{2\ln(2)} = 4, \qquad \mathcal{A}_0(\sigma_*, \varphi_*) = 4 \cdot \frac{1}{4} = 1 = \mathcal{B} \neq 0.$$

(H2) Strict monotonicity of the modulus gap. The map

$$\sigma \mapsto |\mathcal{A}_0(\sigma, \varphi_*)| - |\mathcal{B}| = e^{-2(\sigma-\sigma_*)} - 1$$

is strictly decreasing, positive for $\sigma < \sigma_*$, negative for $\sigma > \sigma_*$.

Step 4: the resonant rectangle and boundary estimates.

Fix $\delta_0 \in (0, \delta)$ and, for $\gamma > 0$ small, let

$$k(\gamma) := \left\lfloor \frac{\varphi_*}{\pi\gamma} \right\rfloor, \qquad \Omega_\gamma := [\sigma_* - \delta_0,\ \sigma_* + \delta_0] \times [k(\gamma)\pi\gamma,\ (k(\gamma)+1)\pi\gamma].$$

In the variable $\omega = \varphi/\gamma$, the band is $\omega \in (k(\gamma)\pi, (k(\gamma)+1)\pi)$, of width π, containing $\omega^*(\eta) = (\ln(2)/\eta)^{1/4}$.

Since the band width in φ is $\pi\gamma \to 0$ when $\gamma \to 0^+$, every $(\sigma, \varphi) \in \Omega_\gamma$ satisfies $|\varphi - \varphi_*| < 2\pi\gamma$. Define, for $\sigma \in [\sigma_* - \delta_0, \sigma_* + \delta_0]$,

$$a(\sigma) := e^{-2(\sigma-\sigma_*)} = \mathcal{A}_0(\sigma, \varphi_*)$$

and define the relative error

$$\epsilon_\gamma(\sigma, \varphi) := \frac{\mathcal{A}_\gamma(\sigma, \varphi)}{a(\sigma)} - 1 = \frac{\mathcal{A}_\gamma(\sigma, \varphi) - a(\sigma)}{a(\sigma)}, \tag{5.50}$$

so that, by definition,

$$\mathcal{A}_\gamma(\sigma, \varphi) = a(\sigma)(1 + \epsilon_\gamma(\sigma, \varphi)). \tag{5.51}$$

We claim $\sup_{\Omega_\gamma} |\epsilon_\gamma| \leq C_1 \eta^{1/6}$ explicitly. Write, using $a(\sigma) = \mathcal{A}_0(\sigma, \varphi_*)$,

$$\mathcal{A}_\gamma(\sigma, \varphi) - a(\sigma) = [\mathcal{A}_\gamma(\sigma, \varphi) - \mathcal{A}_0(\sigma, \varphi)] + [\mathcal{A}_0(\sigma, \varphi) - \mathcal{A}_0(\sigma, \varphi_*)].$$

For the first expression, $\mathcal{A}_\gamma - \mathcal{A}_0 = -e^{-2\sigma}(h_\gamma - h_0)$, and $|h_\gamma - h_0| \leq C_2 \eta^{1/6}$ by (5.47) with $e^{-2\sigma} \leq e^{-2\sigma_* + 2\delta_0} =: K_1$ on Ω_γ, this contributes an upper bound $\leq K_1 C_2 \eta^{1/6}$.

For the second term we use the Mean Value Theorem, $|h_0(\varphi) - h_0(\varphi_*)| \leq \sup |h_0'| \cdot |\varphi - \varphi_*| \leq C_3 \cdot 2\pi\gamma =: C_4\gamma$ with $h_0'(\varphi) = 4\varphi^3(e^{-\varphi^4} - 2e^{-2\varphi^4})$ bounded on a fixed neighborhood of φ_*. Multiplying by $e^{-2\sigma} \leq K_1$, this contributes $\leq K_1 C_4 \gamma \leq K_1 C_4 \eta^{1/6}$ using $\gamma = \eta^{1/4} \leq \eta^{1/6}$ for $\eta \leq 1$.

Combining and dividing by $a(\sigma) \geq e^{-2\delta_0}$, its minimum on Ω_γ,

$$\sup_{\Omega_\gamma} |\epsilon_\gamma| \ \leq\ \frac{K_1(C_2 + C_4)\eta^{1/6}}{e^{-2\delta_0}} \ =:\ C_1\, \eta^{1/6}, \qquad C_1 := 4e^{4\delta_0}(C_2 + C_4). \tag{5.52}$$

Now since the error is bounded, we claim $\mathcal{H}_\gamma \neq 0$ on $\partial\Omega_\gamma$ for γ small.

(a) Horizontal sides. Fixing $\varphi = k\pi\gamma$ or $\varphi = (k+1)\pi\gamma$, the phase equals $e^{-2ik\pi} = 1$ on both sides. By (5.49) with $\cos\theta = 1, \sin\theta = 0$, we get for the first coordinate of (5.49) that $(\mathcal{H}_\gamma)_1 = P_\gamma + 1 = \Re(\mathcal{A}_\gamma) + 1$.

By (5.51), $\Re(\mathcal{A}_\gamma) = a(\sigma)(1 + \Re(\epsilon_\gamma)) \geq a(\sigma)(1 - |\epsilon_\gamma|) \geq e^{-2\delta_0}(1 - C_1\eta^{1/6})$. For $\eta \leq \eta_0$ with $C_1\eta_0^{1/6} \leq \frac{1}{2}$:

$$(\mathcal{H}_\gamma)_1 \geq e^{-2\delta_0} \cdot \tfrac{1}{2} + 1 > 1 \neq 0.$$

Since one coordinate of $\mathcal{H}_\gamma$ is already nonzero, $\mathcal{H}_\gamma \neq (0,0)$ on the horizontal sides.

(b) Vertical sides $\sigma = \sigma_* \pm \delta_0$. With $c(\delta_0) := \frac{1}{2}\min(1 - e^{-2\delta_0}, e^{2\delta_0} - 1) > 0$, it follows for η small enough that $C_1\eta^{1/6} \leq \frac{1}{2}c(\delta_0)e^{-2\delta_0}$.

1. on $\{\sigma = \sigma_* + \delta_0\}$, $|\mathcal{A}_\gamma| \leq e^{-2\delta_0}(1 + C_1\eta^{1/6}) < 1 - c(\delta_0)$,
2. on $\{\sigma = \sigma_* - \delta_0\}$, $|\mathcal{A}_\gamma| \geq e^{2\delta_0}(1 - C_1\eta^{1/6}) > 1 + c(\delta_0)$.

By the reverse triangle inequality,

$$|\mathcal{H}_\gamma| = |e^{-2i\varphi/\gamma}\mathcal{A}_\gamma + 1| \geq ||\mathcal{A}_\gamma| - 1| > c(\delta_0) > 0.$$

This proves the claim.

Step 5: the model map and its degree. Define the model map where the amplitude is frozen at its $\gamma = 0$ profile, while maintaining the same phase,

$$\mathcal{M}_\gamma(\sigma, \varphi) := \Psi\Big(e^{-2i\varphi/\gamma}\, a(\sigma) + 1\Big), \qquad a(\sigma) := e^{-2(\sigma - \sigma_*)}.$$

We seek (σ, φ) with $e^{-2i\varphi/\gamma}a(\sigma) = -1$. Since $a(\sigma) > 0$ is real for every σ and $|e^{-2i\varphi/\gamma}| = 1$, taking moduli gives $a(\sigma) = 1$, i.e. $\sigma = \sigma_*$. With $\sigma = \sigma_*$ fixed, the equation reduces to

$$e^{-2i\omega} = -1, \qquad \omega := \frac{\varphi}{\gamma}. \tag{5.53}$$

By Euler's formula,

$$\cos(2\omega) = -1, \qquad \sin(2\omega) = 0.$$

The equation $\cos\theta = -1$ has solution set exactly $\{\pi + 2\pi k : k \in \mathbb{Z}\}$ with $\theta = 2\omega$, this gives $\omega = (k + \frac{1}{2})\pi$, $k \in \mathbb{Z}$. This is automatically compatible with $\sin(2\omega) = 0$. Hence the full solution set of (5.53) is

$$\omega = \left(k + \frac{1}{2}\right)\pi, \quad k \in \mathbb{Z}. \tag{5.54}$$

This is the phase quantization, within the band $\omega \in (k(\gamma)\pi, (k(\gamma)+1)\pi)$ there is exactly one such point, namely $\omega = (k(\gamma) + \frac{1}{2})\pi$, giving by Proposition 2.17,

$$P_* = \Big(\sigma_*,\ (k(\gamma) + \tfrac{1}{2})\pi\gamma\Big) \in \Omega_\gamma^\circ.$$

With this information we look for the Jacobian of $\mathcal{M}_\gamma$ and its sign. Writing $\mathcal{M}_\gamma = (M_1, M_2)$ with, by (5.49) specialized to $P_\gamma = a(\sigma)$, $Q_\gamma = 0$,

$$M_1(\sigma, \varphi) = a(\sigma)\cos\Big(\frac{2\varphi}{\gamma}\Big) + 1, \qquad M_2(\sigma, \varphi) = -a(\sigma)\sin\Big(\frac{2\varphi}{\gamma}\Big),$$

and using $a'(\sigma) = -2a(\sigma)$:

$$\frac{\partial M_1}{\partial \sigma} = a'(\sigma)\cos\left(\frac{2\varphi}{\gamma}\right), \qquad \frac{\partial M_1}{\partial \varphi} = -\frac{2}{\gamma}a(\sigma)\sin\left(\frac{2\varphi}{\gamma}\right),$$

$$\frac{\partial M_2}{\partial \sigma} = -a'(\sigma)\sin\left(\frac{2\varphi}{\gamma}\right), \qquad \frac{\partial M_2}{\partial \varphi} = -\frac{2}{\gamma}a(\sigma)\cos\left(\frac{2\varphi}{\gamma}\right).$$

At P_* we have $a(\sigma_*) = 1$, $a'(\sigma_*) = -2$, and $\frac{2\varphi}{\gamma}\Big|_{P_*} = 2(k(\gamma) + \frac{1}{2})\pi = (2k(\gamma) + 1)\pi$, an odd multiple of π, where $\cos(\theta) = -1$, $\sin(\theta) = 0$ as in (5.54). Substituting yields

$$\frac{\partial M_1}{\partial \sigma}\bigg|_{P_*} = (-2)(-1) = 2, \qquad \frac{\partial M_1}{\partial \varphi}\bigg|_{P_*} = -\frac{2}{\gamma}(1)(0) = 0,$$

$$\frac{\partial M_2}{\partial \sigma}\bigg|_{P_*} = -(-2)(0) = 0, \qquad \frac{\partial M_2}{\partial \varphi}\bigg|_{P_*} = -\frac{2}{\gamma}(1)(-1) = \frac{2}{\gamma}.$$

Hence the Jacobian matrix at P_* is diagonal:

$$J(P_*) = \begin{pmatrix} 2 & 0 \\ 0 & 2/\gamma \end{pmatrix}, \qquad \det J(P_*) = 2 \cdot \frac{2}{\gamma} - 0 \cdot 0 = \frac{4}{\gamma} > 0.$$

Since P_* is the unique zero of $\mathcal{M}_\gamma$ in Ω_γ° and it is non-degenerate,

$$\deg\left(\mathcal{M}_\gamma, \Omega_\gamma^\circ, 0\right) = \operatorname{sign}\left(\det J(P_*)\right) = +1.$$

Step 6: homotopy and transfer of the degree.

Consider the amplitude homotopy

$$\mathcal{H}_\gamma^{(\mu)}(\sigma, \varphi) := \Psi\Big(e^{-2i\varphi/\gamma}\Big[(1-\mu)\,a(\sigma) + \mu\,\mathcal{A}_\gamma(\sigma, \varphi)\Big] + 1\Big), \qquad \mu \in [0, 1],$$

connecting $\mathcal{H}_\gamma^{(0)} = \mathcal{M}_\gamma$ to $\mathcal{H}_\gamma^{(1)} = \mathcal{H}_\gamma$. The interpolated amplitude $\mathcal{A}^{(\mu)} := (1-\mu)a + \mu\mathcal{A}_\gamma$ satisfies the same boundary bounds.

1. on the horizontal sides, $\Re(\mathcal{A}^{(\mu)}) \geq (1-\mu)e^{-2\delta_0} + \mu\, e^{-2\delta_0}(1 - C_1\eta^{1/6}) > 0$, so $(\mathcal{H}_\gamma^{(\mu)})_1 \geq 1 - o(1) > \frac{1}{2}$,
2. on the vertical sides, by (5.51) both $a(\sigma)$ and $|\mathcal{A}_\gamma(\sigma, \varphi)|$ lie strictly on the same side of 1 since both satisfy $< 1 - c(\delta_0)$ at $\sigma_* + \delta_0$, and both satisfy $> 1 + c(\delta_0)$ at $\sigma_* - \delta_0$, for γ small, preserved due to the convex combination involved. Whence $|\mathcal{H}_\gamma^{(\mu)}| \geq ||\mathcal{A}^{(\mu)}| - 1| > c(\delta_0)/2 > 0$.

Thus $\mathcal{H}_\gamma^{(\mu)} \neq 0$ on $\partial\Omega_\gamma$ for every $\mu \in [0, 1]$, and by homotopy invariance of the topological degree Proposition 2.16,

$$\deg\left(\mathcal{H}_\gamma, \Omega_\gamma^\circ, 0\right) = \deg\left(\mathcal{M}_\gamma, \Omega_\gamma^\circ, 0\right) = +1 \neq 0.$$

By the existence property of the degree, $\mathcal{H}_\gamma$ has at least one zero $(\tilde{\sigma}, \tilde{\varphi}) \in \Omega_\gamma^\circ$.

To conclude we undo the change of variables $\tilde{s} = \tilde{\sigma} + i\tilde{\varphi}/\gamma \in \mathcal{S}_\eta$. We know that a zero of $\Psi^{-1} \circ \mathcal{H}_\gamma$ is exactly a zero of the exact characteristic function $\mathcal{G}$ with

$$\tilde{\sigma} \in (-\ln(2) - \delta_0,\ -\ln(2) + \delta_0), \qquad \eta\,\tilde{\omega}^4 = \tilde{\varphi}^4 \in \big((k(\gamma)\pi\gamma)^4, ((k(\gamma)+1)\pi\gamma)^4\big) \xrightarrow[\gamma \to 0^+]{} \varphi_*^4 = \ln(2).$$

Hence for every $\gamma \in (0, \gamma_1]$ we have $\mathcal{G}(\gamma^4, \tilde{s}) = 0$, which means in our original variables that for every $\eta \in (0, \eta_1^4]$, there exists $\tilde{s} \in S_\eta$ such that $\Re(\tilde{s}) \in (-\ln(2) - \delta_0, -\ln(2) + \delta_0)$. This finally gives

$$-\ln(2) - \delta_0 < \Re(\tilde{s}) < \sigma_\eta.$$

Choosing $\delta_0 < \delta$ and $\tilde{\eta} = \min\{\eta_1, \gamma_1^4\}$ gives the result. □

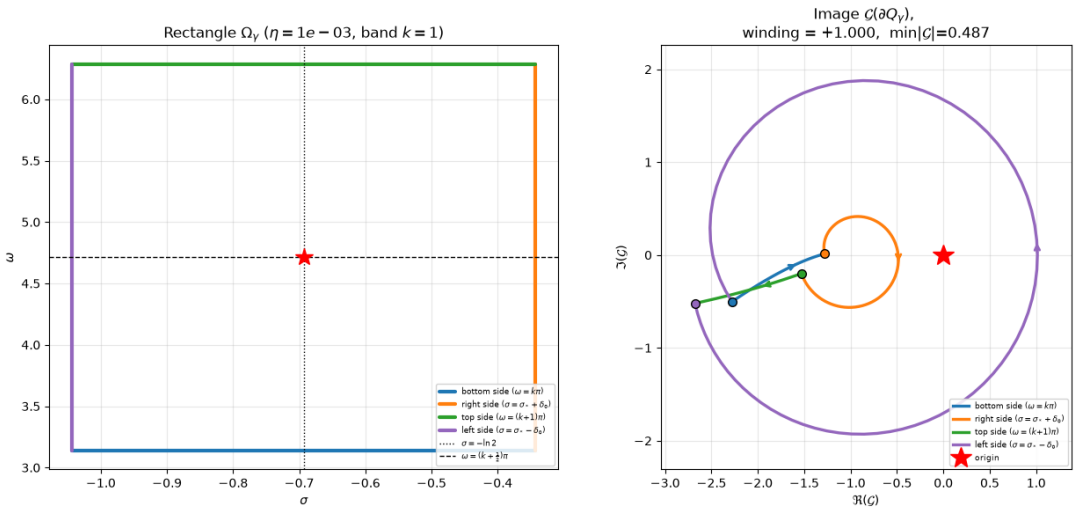


Figure 5.5: Image of the rectangle Ω_γ under $\mathcal{G}$

Chapter 6

Conclusions and further work

6.1 Conclusions

Throughout this manuscript we studied the robustness of delayed output-feedback stabilization for 2×2 hyperbolic systems in which the control and the observation are anti-located. The starting point, recalled in Chapter 2, is that such feedback laws are designed for the inviscid transport model and are, in general, arbitrarily sensitive to small errors in the transport velocity: an infinitesimal mismatch between the delay used by the controller and the true transport time can destroy exponential stability. The unifying theme of Chapters 3, 4 and 5 is that this apparent fragility is an artefact of neglecting a small, physically present dissipative term: once such a term is reinstated, however small, the very same delayed feedback law becomes robust, and the maximal spectral abscissa of the closed-loop system converges, as the dissipation parameter $\eta \to 0^+$, to the explicit value $-\ln(2)$, up to the relevant time and velocity scales. In each chapter this phenomenon is established by combining three tools developed in Chapter 2: the transfer function and Laplace domain description of the closed-loop characteristic equation, an asymptotic analysis of its roots as $\eta \to 0^+$, and, where an explicit change of variables is unavailable, topological degree theory to locate a root without solving for it explicitly. Concretely,

1. We present in Chapter 3, the results in [BCH23] on stability and robustness for a hyperbolic system with boundary coupling and a second-order diffusive perturbation, by compactness arguments exploiting the plant form of the system. There, the authors only established the upper bound on the maximal spectral abscissa, together with a complementary destabilizing example, see Section 3.6, showing that the same regularization mechanism can fail when the diffusion acts differently relative to the boundary coupling.

2. We present in Chapter 4, the results in [BCH25b] on stability and robustness for a hyperbolic system with internal coupling and a second-order diffusive perturbation, again by compactness arguments, to obtain the upper bound on the maximal spectral abscissa. In this chapter the authors went further and also proved the matching lower bound, so that the optimality of the stabilization constant $\ln(2)$ is established rigorously rather than only conjectured. This was achieved through a topological degree argument that locates a pole of the closed-loop characteristic equation near the numerically conjectured spectral abscissa.

3. We originally address in Chapter 5 the stability and robustness with respect to fourth-order perturbations for a hyperbolic system with boundary coupling. Since the fourth-order characteristic equation no longer admits an explicit solution formula, we developed an asymptotic classification of its four roots as $\eta \to 0^+$, one finite branch and three exploding branches, and used it to build a first-order approximation of the transfer function with an explicit error bound. Combining this expansion with topological degree arguments, we showed that $\ln(2)$ remains the optimal stabilization constant even under this higher-order perturbation, and we complemented

the analytical results with numerical implementations illustrating the behavior of the spectrum and the geometric picture underlying the topological degree argument.

Taken together, these three results indicate that the robustifying effect of a small dissipative term, and the value ln(2) of the associated optimal stabilization constant, are not tied to the specific second-order diffusive mechanism considered in [BCH23; BCH25b]: the same qualitative picture survives a genuinely different, higher-order perturbation and a different coupling structure, even though the underlying complex analytic argument has to be substantially reworked in the fourth-order case. The counterexample of Section 3.6, however, is a reminder that this robustness is not automatic: it depends delicately on where, and how, the perturbation is coupled to the boundary conditions, and understanding the precise boundary between the stabilizing and destabilizing regimes remains only partially understood.

6.2 Further work

We think that there are many possible generalizations of the situation studied in this thesis.

1. In Chapter 4 we only modify one of the equations, taking into account that now we have internal coupling, which differs from the situation on Chapter 3. Is it possible to show the stability and robustness with respect to fourth-order perturbation for a hyperbolic system with internal coupling, following the previous idea?

2. What happens if we consider a perturbation of odd order? Do we still have a stabilization result that is, or maybe is not, robust with respect to perturbations of the transport velocity?

3. In the same line, is the optimal constant ln(2) intrinsic to the transport equations, or does it depend on the order of the perturbation? Chapter 3 and Chapter 5 show that the optimal constant coincides regardless of the order of the perturbation. Is this still the case when we tackle the fourth-order perturbed system with internal coupling?

4. In this work we considered the same constant velocity on both equations from Chapter 3. Is it possible to consider a variable velocity $v(x)$ and achieve a stabilization result depending on the values of $v(x)$?

5. Along the same way, the key properties of the systems in this thesis seems to be the convergence $f_\eta(s) \to f_0(s) = e^{-s}$ and the delayed feedback mechanism $U(t) = -Y(t-1)$, which in frequency space yields an exponential e^{-s}. Qualitatively this implies the existence of infinite poles for the characteristic equation once we close the loop of the system since the delay mechanism and original transfer function f_0 yields the same transcendental function. Is it possible to find a different feedback mechanism for another hyperbolic system to obtain the same property?

6. Recently, in [BCH25a], Bastin, Coron and Hayat reported a new result on the necessary and sufficient robustness condition for the stabilization of 2×2 hyperbolic systems in the presence of internal and boundary coupling. The sufficient condition is achieved by backstepping methods, and the novelty is the necessary condition, proved by topological degree arguments. The first question is to explore further connections between backstepping and topological degree theory, and the second one is whether it is possible to incorporate higher-order perturbations into the problem and employ the transfer function approach instead of the backstepping process.

Chapter A

Appendix

Here we collect the used codes[1] on this thesis.

A.1 Codes for Chapter 3

Code 1

```
  """
char_eq_utils.py
=================

Utilities shared by the scripts `chapter2_diffusion_robustness.py`
and `chapter3_viscosity_fluid.py`.

"""

from __future__ import annotations

import itertools
import numpy as np
import mpmath as mp
import matplotlib
import matplotlib.pyplot as plt

# Privileged line -ln(2)
LN2 = float(np.log(2.0))

# Root search

def find_spectrum(
    h,
    re_range=(-6.0, 1.0),
    im_range=(0.0, 40.0),
    n_re=25,
    n_im=25,
    dps=30,
    residual_tol=1e-8,
    dedupe_tol=5e-3,
    mirror_conjugates=True,
    maxsteps=60,
    verbose=False,
):
    """
    Numerically searches for the roots of h(s) = 0 in the
    box  Re(s) in re_range,  Im(s) in im_range, using a grid of
    starting points
    """
```

[1] Claude was used to improve the accuracy of the numerical plots in this work. In particular, it proposed the use of Müller's iteration method to make the underlying code more efficient, and suggested an improved structure for generating the winding-number plots.

```
old_dps = mp.mp.dps
mp.mp.dps = dps
roots = []

# Generous but finite bound for |Re(s)|, |Im(s)| during the Muller
# iterations: if a candidate shoots outside this region, it is
# discarded immediately instead of continuing to evaluate
# ever-larger exponentials (which in practice could leave the
# computation "stuck" indefinitely).
re_bound = 4.0 * (re_range[1] - re_range[0]) + 50.0
im_bound = 4.0 * (im_range[1] - max(im_range[0], 0.0)) + 200.0

def bounded_h(s):
    if abs(s.real) > re_bound or abs(s.imag) > im_bound:
        raise ValueError("Muller iteration outside domain (discarded)")
    return h(s)

try:
    re_vals = np.linspace(re_range[0], re_range[1], n_re)
    im_lo = 0.0 if mirror_conjugates else im_range[0]
    im_vals = np.linspace(im_lo, im_range[1], n_im)

    for sigma, omega in itertools.product(re_vals, im_vals):
        s0 = mp.mpc(float(sigma), float(omega))
        try:
            root = mp.findroot(bounded_h, s0, solver="muller", tol=1e-
                                                    25,
                               maxsteps=maxsteps)
        except Exception:
            continue

        try:
            resid = abs(h(root))
        except Exception:
            continue
        if resid > residual_tol:
            continue

        # Discard roots outside the search box
        re_margin = 0.5 * (re_range[1] - re_range[0]) + 1.0
        im_margin = 0.5 * (im_range[1] - im_range[0]) + 5.0
        if not (re_range[0] - re_margin <= root.real <= re_range[1] +
                                                re_margin):
            continue
        if not (im_range[0] - im_margin <= root.imag <= im_range[1] +
                                                im_margin):
            continue

        c = complex(root)
        is_dup = any(abs(c - r) < dedupe_tol for r in roots)
        if not is_dup:
            roots.append(c)

finally:
    mp.mp.dps = old_dps

if mirror_conjugates:
    mirrored = []
    for r in roots:
        mirrored.append(r)
        if abs(r.imag) > dedupe_tol:
            mirrored.append(complex(r.real, -r.imag))
```

```
        roots = mirrored
        unique = []
        for r in roots:
            if not any(abs(r - u) < dedupe_tol for u in unique):
                unique.append(r)
        roots = unique

    roots.sort(key=lambda z: -z.real)

    if verbose:
        print(f"  -> {len(roots)} roots found.")
        if roots:
            print(f"     spectral abscissa (max Re(s)) = {roots[0].real:.6f
                                                        }")

    return roots


# Plotting the spectrum

def plot_spectrum_overlay(
    h_factory,
    param_list,
    re_range=(-6.0, 1.0),
    im_range=(0.0, 40.0),
    n_re=25,
    n_im=25,
    dps=30,
    ax=None,
    mark_ln2=True,
    title="Spectrum (poles) of the characteristic equation",
    verbose=True,
):
    """
    Overlays, on the same plot, the spectrum of the system
    for several parameter combinations, marking the line Re(s) = -ln(2).

    """
    if ax is None:
        fig, ax = plt.subplots(figsize=(7.5, 6))
    else:
        fig = ax.figure

    cmap = matplotlib.colormaps["viridis"].resampled(max(len(param_list), 2
                                                         ))
    all_roots_info = []

    for idx, (eta, eps) in enumerate(param_list):
        if verbose:
            print(f"Searching spectrum for eta={eta}, eps={eps} ...")
        h = h_factory(eta, eps)
        roots = find_spectrum(
            h,
            re_range=re_range,
            im_range=im_range,
            n_re=n_re,
            n_im=n_im,
            dps=dps,
            verbose=verbose,
        )
        all_roots_info.append((eta, eps, roots))
        if not roots:
```

```
            continue
        re_parts = [r.real for r in roots]
        im_parts = [r.imag for r in roots]
        color = cmap(idx)
        label = f"$\\eta={eta:g}$" + (f", $\\varepsilon={eps:g}$" if eps !=
                                                0 else "")
        ax.scatter(re_parts, im_parts, s=32, color=color, label=label,
                   edgecolor="black", linewidth=0.4, zorder=3)

    if mark_ln2:
        ax.axvline(-LN2, color="crimson", linestyle="--", linewidth=1.6,
                   label=r"$\Re(s)=-\ln 2$", zorder=2)

    ax.axvline(0, color="black", linewidth=0.8, zorder=1)
    ax.axhline(0, color="black", linewidth=0.5, alpha=0.5, zorder=1)
    ax.set_xlabel(r"$\Re(s)$")
    ax.set_ylabel(r"$\Im(s)$")
    ax.set_title(title)
    ax.legend(loc="best", fontsize=9)
    ax.grid(alpha=0.3)
    fig.tight_layout()
    return fig, ax, all_roots_info


# Plotting transfer functions

def plot_transfer_function_overlay(
    tf_factory,
    param_list,
    omega_range=(1e-3, 50.0),
    n_points=2000,
    log_omega=True,
    magnitude_db=False,
    ax=None,
    title="Transfer function",
    ylabel=None,
):
    """
    Overlays, on the same plot, the magnitude |F(i*omega)| of a
    transfer function F for different parameter combinations
    (eta, epsilon).


    """
    if ax is None:
        fig, ax = plt.subplots(figsize=(8, 5))
    else:
        fig = ax.figure

    if log_omega:
        omegas = np.logspace(np.log10(omega_range[0]), np.log10(omega_range
                                                [1]), n_points)
    else:
        omegas = np.linspace(omega_range[0], omega_range[1], n_points)

    s_vals = 1j * omegas
    cmap = matplotlib.colormaps["plasma"].resampled(max(len(param_list), 2)
                                            )

    for idx, (eta, eps) in enumerate(param_list):
        F = tf_factory(eta, eps)
        with np.errstate(all="ignore"):
```

```
            vals = F(s_vals)
        mag = np.abs(vals)
        if magnitude_db:
            with np.errstate(divide="ignore"):
                mag = 20.0 * np.log10(mag)
        color = cmap(idx)
        label = f"$\\eta={eta:g}$" + (f", $\\varepsilon={eps:g}$" if eps !=
                                                    0 else "")
        ax.plot(omegas, mag, color=color, label=label, linewidth=1.6)

    if log_omega:
        ax.set_xscale("log")
    ax.set_xlabel(r"$\omega$  (with $s=i\omega$)")
    ax.set_ylabel(ylabel or (r"$|F(i\omega)|$ [dB]" if magnitude_db else r"
                                        $|F(i\omega)|$"))
    ax.set_title(title)
    ax.legend(loc="best", fontsize=9)
    ax.grid(alpha=0.3, which="both")
    fig.tight_layout()
    return fig, ax
```

Code 2

```
  #!/usr/bin/env python3
# -*- coding: utf-8 -*-
"""
chapter2_diffusion_robustness.py

"""

from __future__ import annotations

import argparse
import os
import sys

import numpy as np
import mpmath as mp

import matplotlib
if os.environ.get("DISPLAY", "") == "" and sys.platform.startswith("linux")
                                        :
    matplotlib.use("Agg")
import matplotlib.pyplot as plt

from char_eq_utils import (
    LN2,
    plot_spectrum_overlay,
    plot_transfer_function_overlay,
)

COMANDO = "spectrum"

ETA_1, ETA_2, ETA_3 = 0.02, 0.02, 0.0
EPS_1, EPS_2, EPS_3 = 0.0, 0.1, 0.1

def lambda12_mp(s, eta, eps):

    eta = mp.mpf(eta)
    eps = mp.mpf(eps)
    z = mp.sqrt((1 + eps) ** 2 + 4 * eta * s)
    lam1 = ((1 + eps) + z) / (2 * eta)
```

```
    lam2 = ((1 + eps) - z) / (2 * eta)
    return lam1, lam2

def f_eta_eps_mp(s, eta, eps):
    """Transfer function of a single channel, f_{eta,eps}(s)."""
    if eta == 0:
        return mp.exp(-s / (1 + eps))
    lam1, lam2 = lambda12_mp(s, eta, eps)
    num = lam1 - lam2
    den = lam1 * mp.exp(-lam2) - lam2 * mp.exp(-lam1)
    return num / den

def char_closed_loop_mp_factory(eta, eps):
    """h(s) = f^2(s) - f(s) e^{-s} - 1   (closed loop, U(t) = -Y(t-1))."""
    def h(s):
        f = f_eta_eps_mp(s, eta, eps)
        return f * f - f * mp.exp(-s) - 1
    return h

def char_open_loop_mp_factory(eta, eps):
    """h(s) = f^2(s) - 1   (open loop)."""
    def h(s):
        f = f_eta_eps_mp(s, eta, eps)
        return f * f - 1
    return h

def lambda12_np(s, eta, eps):
    z = np.sqrt((1 + eps) ** 2 + 4 * eta * s + 0j)
    lam1 = ((1 + eps) + z) / (2 * eta)
    lam2 = ((1 + eps) - z) / (2 * eta)
    return lam1, lam2

def f_eta_eps_np(s, eta, eps):
    if eta == 0:
        return np.exp(-s / (1 + eps))
    lam1, lam2 = lambda12_np(s, eta, eps)
    num = lam1 - lam2
    den = lam1 * np.exp(-lam2) - lam2 * np.exp(-lam1)
    return num / den

def G_open_loop_np(s, eta, eps):
    f = f_eta_eps_np(s, eta, eps)
    return f / (1.0 - f ** 2)

def make_tf_factory(which: str):
    if which == "f":
        def factory(eta, eps):
            return lambda s: f_eta_eps_np(s, eta, eps)
        return factory
    elif which == "G":
        def factory(eta, eps):
            return lambda s: G_open_loop_np(s, eta, eps)
        return factory
    else:
        raise ValueError("which must be 'f' (single channel) or 'G' (full
                                                open loop)")

def _parse_args(argv=None):
    p = argparse.ArgumentParser(
        description="Chapter 2: spectrum and transfer functions "
                    "of the system with diffusion (Bastin-Coron-Hayat)."
    )
```

```
    sub = p.add_subparsers(dest="cmd", required=True)


    ps = sub.add_parser("spectrum", help="Plot the spectrum (poles) in the
                                          complex plane.")
    ps.add_argument("--eta", type=float, nargs="+", default=[0.02, 0.05, 0.
                                          1],
                    help="Values of eta (diffusion) to overlay.")
    ps.add_argument("--eps", type=float, nargs="+", default=[0.0],
                    help="Values of epsilon (velocity perturbation) to
                                                        overlay.")
    ps.add_argument("--mode", choices=["closed", "open"], default="closed",
                    help="Closed-loop characteristic equation (with U(t)=-
                                                        Y(t-1)) "
                         "or open-loop.")
    ps.add_argument("--re-min", type=float, default=-6.0)
    ps.add_argument("--re-max", type=float, default=1.0)
    ps.add_argument("--im-max", type=float, default=30.0)
    ps.add_argument("--n-re", type=int, default=22, help="Search grid
                                          resolution (real part).")
    ps.add_argument("--n-im", type=int, default=22, help="Search grid
                                          resolution (imaginary part).")
    ps.add_argument("--dps", type=int, default=30, help="mpmath precision
                                          digits.")
    ps.add_argument("--output", type=str, default="chapter2_spectrum.png")
    ps.add_argument("--quiet", action="store_true")

    # --- subcommand: transfer ---
    pt = sub.add_parser("transfer", help="Plot the transfer function |F(i*
                                          omega)|.")
    pt.add_argument("--eta", type=float, nargs="+", default=[0.02, 0.05, 0.
                                          1])
    pt.add_argument("--eps", type=float, nargs="+", default=[0.0])
    pt.add_argument("--which", choices=["f", "G"], default="f",
                    help="'f' = single-channel transfer function, "
                         "'G' = full open-loop transfer function f/(1-f^2)
                                                        .")
    pt.add_argument("--omega-min", type=float, default=1e-3)
    pt.add_argument("--omega-max", type=float, default=50.0)
    pt.add_argument("--linear", action="store_true", help="Use linear scale
                                           for omega (default: log).")
    pt.add_argument("--db", action="store_true", help="Plot magnitude in
                                          decibels.")
    pt.add_argument("--output", type=str, default="
                                          chapter2_transfer_function.png")

    return p.parse_args(argv)

def _param_grid(eta_list, eps_list):
    """Ordered cartesian product of (eta, eps), avoiding duplicate
    combinations when one of the lists has a single element."""
    return [(eta, eps) for eta in eta_list for eps in eps_list]

def cmd_spectrum(args, params=None):
    if params is None:
        params = _param_grid(args.eta, args.eps)
    factory = char_closed_loop_mp_factory if args.mode == "closed" else
                                          char_open_loop_mp_factory
    title = ("Closed-loop spectrum"
             if args.mode == "closed" else
             "Open-loop spectrum (with viscosity)")
```

```
    fig, ax, info = plot_spectrum_overlay(
        factory,
        params,
        re_range=(args.re_min, args.re_max),
        im_range=(0.0, args.im_max),
        n_re=args.n_re,
        n_im=args.n_im,
        dps=args.dps,
        title=title,
        verbose=not args.quiet,
    )
    fig.savefig(args.output, dpi=150)
    print(f"\nFigure saved to: {args.output}")

    print("\nSummary (maximum spectral abscissa Re(s) found, compare with -
                                        ln(2) = "
          f"{-LN2:.6f}):")
    for eta, eps, roots in info:
        if roots:
            print(f"  eta={eta:<8g} eps={eps:<8g} ->  max Re(s) = {roots[0
                                                ].real:.6f}")
        else:
            print(f"  eta={eta:<8g} eps={eps:<8g} ->  (no roots found in
                                                the search box)")
    plt.show()

def cmd_transfer(args, params=None):
    factory = make_tf_factory(args.which)
    if params is None:
        params = _param_grid(args.eta, args.eps)
    label = "f_{\\eta,\\varepsilon}(s)" if args.which == "f" else "G(s)=f/(
                                        1-f^2)"
    fig, ax = plot_transfer_function_overlay(
        factory,
        params,
        omega_range=(args.omega_min, args.omega_max),
        log_omega=not args.linear,
        magnitude_db=args.db,
        title=f"Chapter 2 -- Frequency response of ${label}$",
    )
    fig.savefig(args.output, dpi=150)
    print(f"Figure saved to: {args.output}")
    plt.show()

def _run_from_config():
    params = list(zip([ETA_1, ETA_2, ETA_3], [EPS_1, EPS_2, EPS_3]))
    print("(No command-line arguments: using the configuration "
          f"from the file -> command='{COMANDO}'")
    print(f" (eta, eps) = {params})\n")

    args = _parse_args([COMANDO])
    if COMANDO == "spectrum":
        cmd_spectrum(args, params=params)
    elif COMANDO == "transfer":
        cmd_transfer(args, params=params)
    else:
        raise SystemExit(f"COMANDO must be 'spectrum' or 'transfer', not {
                                            COMANDO!r}")

def main(argv=None):
    if argv is None and len(sys.argv) == 1:
        _run_from_config()
```

```
        return
    args = _parse_args(argv)
    if args.cmd == "spectrum":
        cmd_spectrum(args)
    elif args.cmd == "transfer":
        cmd_transfer(args)

if __name__ == "__main__":
    main()
```

Code 3

```
  #!/usr/bin/env python3
# -*- coding: utf-8 -*-
"""
section2_6_destabilization.py
================================

"""

from __future__ import annotations

import argparse
import os
import sys

import numpy as np
import mpmath as mp

import matplotlib
if os.environ.get("DISPLAY", "") == "" and sys.platform.startswith("linux")
                                        :
    matplotlib.use("Agg")
import matplotlib.pyplot as plt

from char_eq_utils import find_spectrum
from chapter2_diffusion_robustness import f_eta_eps_mp, f_eta_eps_np

COMANDO = "spectrum"

ETA_1, ETA_2, ETA_3 = 0.1, 0.0, 0.0

def h_mp_factory(eta):
    def h(s):
        f = f_eta_eps_mp(s, eta, 0.0)
        return f - 1 - mp.exp(-s)
    return h

def h_np(s, eta):
    f = f_eta_eps_np(s, eta, 0.0)
    return f - 1 - np.exp(-s)

def _parse_args(argv=None):
    p = argparse.ArgumentParser(
        description="Simplified system and its characteristic "
                    "equation f_eta(s) = 1+e^{-s}."
    )
    sub = p.add_subparsers(dest="cmd", required=True)

    ps = sub.add_parser("spectrum", help="Spectrum (poles) in the complex
                                          plane.")
```

```
    ps.add_argument("--eta", type=float, nargs="+", default=[0.05, 0.02, 0.
                                        01])
    ps.add_argument("--re-min", type=float, default=-3.0)
    ps.add_argument("--re-max", type=float, default=0.4)
    ps.add_argument("--im-max", type=float, default=40.0)
    ps.add_argument("--n-re", type=int, default=16)
    ps.add_argument("--n-im", type=int, default=26)
    ps.add_argument("--dps", type=int, default=30)
    ps.add_argument("--output", type=str, default="section2_6_spectrum.png"
                                        )

    pw = sub.add_parser("sweep", help="Sweep over eta -> reports sigma_eta
                                        (trend towards 0).")
    pw.add_argument("--eta", type=float, nargs="+",
                    default=[0.5, 0.2, 0.1, 0.05, 0.02, 0.01, 0.005, 0.002
                                                        ])
    pw.add_argument("--re-min", type=float, default=-3.0)
    pw.add_argument("--re-max", type=float, default=0.4)
    pw.add_argument("--im-max", type=float, default=60.0)
    pw.add_argument("--n-re", type=int, default=16)
    pw.add_argument("--n-im", type=int, default=40)
    pw.add_argument("--dps", type=int, default=30)
    pw.add_argument("--output", type=str, default="section2_6_sweep.png")

    return p.parse_args(argv)

def cmd_spectrum(args, eta_list=None):
    if eta_list is None:
        eta_list = args.eta
    fig, ax = plt.subplots(figsize=(7.5, 6))
    cmap = matplotlib.colormaps["viridis"].resampled(max(len(eta_list), 2))
    for idx, eta in enumerate(eta_list):
        print(f"Searching spectrum for eta={eta} ...")
        roots = find_spectrum(
            h_mp_factory(eta),
            re_range=(args.re_min, args.re_max),
            im_range=(0.0, args.im_max),
            n_re=args.n_re, n_im=args.n_im, dps=args.dps, verbose=True,
        )
        if not roots:
            continue
        re_parts = [r.real for r in roots]
        im_parts = [r.imag for r in roots]
        color = cmap(idx)
        ax.scatter(re_parts, im_parts, s=30, color=color,
                   label=fr"$\eta={eta:g}$ ($\max\Re(s)={roots[0].real:.4f}
                                                        $)",
                   edgecolor="black", linewidth=0.3, zorder=3)

    ax.axvline(0, color="crimson", linestyle="--", linewidth=1.6,
               label=r"$\Re(s)=0$ (marginal stability)", zorder=2)
    ax.axhline(0, color="black", linewidth=0.5, alpha=0.4)
    ax.set_xlabel(r"$\Re(s)$")
    ax.set_ylabel(r"$\Im(s)$")
    ax.set_title("Spectrum of $f_\\eta(s)=1+e^{-s}$")
    ax.legend(loc="best", fontsize=8)
    ax.grid(alpha=0.3)
    fig.tight_layout()
    fig.savefig(args.output, dpi=150)
    print(f"\nFigure saved to: {args.output}")
    plt.show()
```

```
def cmd_sweep(args, eta_list=None):
    if eta_list is None:
        eta_list = args.eta
    etas, sigmas = [], []
    for eta in eta_list:
        print(f"eta={eta} ...", end=" ", flush=True)
        roots = find_spectrum(
            h_mp_factory(eta),
            re_range=(args.re_min, args.re_max),
            im_range=(0.0, args.im_max),
            n_re=args.n_re, n_im=args.n_im, dps=args.dps, verbose=False,
        )
        if roots:
            sigma = roots[0].real
            print(f"sigma_eta ~ {sigma:.6f}")
            etas.append(eta)
            sigmas.append(sigma)
        else:
            print("no roots found")

    fig, ax = plt.subplots(figsize=(7, 5))
    ax.axhline(0, color="crimson", linestyle="--", linewidth=1.5, label=r"$
                                            \Re(s)=0$")
    ax.plot(etas, sigmas, "o-", color="navy")
    ax.set_xscale("log")
    ax.set_xlabel(r"$\eta$")
    ax.set_ylabel(r"$\sigma_\eta$ (estimated)")
    ax.set_title("Section 2.6 âĂŤ trend of $\\sigma_\\eta$ as $\\eta\\to0^+
                                            $\n"
                 "(Conjecture 2: $\\sigma_\\eta$ approaches $0$)")
    ax.legend()
    ax.grid(alpha=0.3, which="both")
    fig.tight_layout()
    fig.savefig(args.output, dpi=150)
    print(f"\nFigure saved to: {args.output}")
    print("\nNote: sigma_eta reported is the maximum Re(s) found within the
                                             search\n"
          "box (Im(s) up to --im-max); for very small eta the dominant pole
                                              may sit\n"
          "at large |Im(s)|, so consider increasing --im-max if the
                                             reported value\n"
          "does not seem to have stabilized when widening the search.")
    plt.show()

def _run_from_config():
    eta_list = [ETA_1, ETA_2, ETA_3]
    print("(No command-line arguments: using the configuration "
          f"from the file -> command='{COMANDO}'")
    print(f" eta = {eta_list})\n")

    args = _parse_args([COMANDO])
    if COMANDO == "spectrum":
        cmd_spectrum(args, eta_list=eta_list)
    elif COMANDO == "sweep":
        cmd_sweep(args, eta_list=eta_list)
    else:
        raise SystemExit(f"COMANDO must be 'spectrum' or 'sweep', not {
                                               COMANDO!r}")

def main(argv=None):
    if argv is None and len(sys.argv) == 1:
        _run_from_config()
```

```
        return
    args = _parse_args(argv)
    if args.cmd == "spectrum":
        cmd_spectrum(args)
    elif args.cmd == "sweep":
        cmd_sweep(args)

if __name__ == "__main__":
    main()
```

A.2 Codes for Chapter 4

These scripts also rely on `char_eq_utils.py`, listed above in the codes for Chapter 3, for the shared root-search and plotting utilities.

Code 4

```
  #!/usr/bin/env python3
# -*- coding: utf-8 -*-
"""
chapter3_viscosity_fluid.py

"""

from __future__ import annotations

import argparse
import os
import sys

import numpy as np
import mpmath as mp

import matplotlib
if os.environ.get("DISPLAY", "") == "" and sys.platform.startswith("linux")
                                        :
    matplotlib.use("Agg")
import matplotlib.pyplot as plt

from char_eq_utils import (
    LN2,
    plot_spectrum_overlay,
    plot_transfer_function_overlay,
)

COMANDO = "spectrum"

ETA_1, ETA_2, ETA_3 = 0.01, 0.0, 0.01
EPS_1, EPS_2, EPS_3 = 0.1, 0.1, 0.0

def lambda_mp(s, eta, eps):
    eta = mp.mpf(eta)
    eps = mp.mpf(eps)
    return s / mp.sqrt(1 + eps + eta * s)

def G_open_loop_mp(s, eta, eps):
    """Open-loop transfer function."""
    eta_ = mp.mpf(eta)
    eps_ = mp.mpf(eps)
    lam = lambda_mp(s, eta, eps)
```

```
    return 2 * mp.exp(lam) / (mp.sqrt(1 + eps_ + eta_ * s) * (mp.exp(2 *
                                            lam) - 1))

def char_closed_loop_mp_factory(eta, eps):
    """
    h(s) = G_eps(eta, s)
         = e^s sqrt(1+eps+eta s) ( e^{s/sqrt(1+eps+eta s)} - e^{-s/sqrt(1+
                                                eps+eta s)} ) + 1

    with eps=0 the equation G(eta,s)=0 is recovered.
    """
    eta_ = mp.mpf(eta)
    eps_ = mp.mpf(eps)

    def h(s):
        w = mp.sqrt(1 + eps_ + eta_ * s)
        lam = s / w
        return mp.e ** s * w * (mp.exp(lam) - mp.exp(-lam)) + 1

    return h

def lambda_np(s, eta, eps):
    return s / np.sqrt(1 + eps + eta * s + 0j)

def G_open_loop_np(s, eta, eps):
    lam = lambda_np(s, eta, eps)
    w = np.sqrt(1 + eps + eta * s + 0j)
    return 2 * np.exp(lam) / (w * (np.exp(2 * lam) - 1))

def make_tf_factory():
    def factory(eta, eps):
        return lambda s: G_open_loop_np(s, eta, eps)
    return factory

def _parse_args(argv=None):
    p = argparse.ArgumentParser(
        description="Chapter 3: spectrum and transfer function of the "
                    "fluid flow system with viscosity (Bastin-Coron-Hayat)
                                                            ."
    )
    sub = p.add_subparsers(dest="cmd", required=True)

    ps = sub.add_parser("spectrum", help="Plot the closed-loop spectrum (
                                            poles).")
    ps.add_argument("--eta", type=float, nargs="+", default=[0.02, 0.05, 0.
                                            1],
                    help="Values of eta (viscosity) to overlay.")
    ps.add_argument("--eps", type=float, nargs="+", default=[0.0],
                    help="Values of epsilon (perturbation c^2=1+eps) to
                                                            overlay.")
    ps.add_argument("--re-min", type=float, default=-6.0)
    ps.add_argument("--re-max", type=float, default=1.0)
    ps.add_argument("--im-max", type=float, default=30.0)
    ps.add_argument("--n-re", type=int, default=22)
    ps.add_argument("--n-im", type=int, default=22)
    ps.add_argument("--dps", type=int, default=30)
    ps.add_argument("--output", type=str, default="chapter3_spectrum.png")
    ps.add_argument("--quiet", action="store_true")

    pt = sub.add_parser("transfer", help="Plot the transfer function |G(i*
                                            omega)|.")
```

```
    pt.add_argument("--eta", type=float, nargs="+", default=[0.02, 0.05, 0.
                                        1])
    pt.add_argument("--eps", type=float, nargs="+", default=[0.0])
    pt.add_argument("--omega-min", type=float, default=1e-3)
    pt.add_argument("--omega-max", type=float, default=50.0)
    pt.add_argument("--linear", action="store_true")
    pt.add_argument("--db", action="store_true")
    pt.add_argument("--output", type=str, default="
                                        chapter3_transfer_function.png")

    return p.parse_args(argv)

def _param_grid(eta_list, eps_list):
    return [(eta, eps) for eta in eta_list for eps in eps_list]

def cmd_spectrum(args, params=None):
    if params is None:
        params = _param_grid(args.eta, args.eps)
    fig, ax, info = plot_spectrum_overlay(
        char_closed_loop_mp_factory,
        params,
        re_range=(args.re_min, args.re_max),
        im_range=(0.0, args.im_max),
        n_re=args.n_re,
        n_im=args.n_im,
        dps=args.dps,
        title="Closed-loop spectrum",
        verbose=not args.quiet,
    )
    fig.savefig(args.output, dpi=150)
    print(f"\nFigure saved to: {args.output}")

    print("\nSummary (maximum spectral abscissa Re(s) found, compare with -
                                        ln(2) = "
          f"{-LN2:.6f}):")
    for eta, eps, roots in info:
        if roots:
            print(f"  eta={eta:<8g} eps={eps:<8g} ->  max Re(s) = {roots[0
                                        ].real:.6f}")
        else:
            print(f"  eta={eta:<8g} eps={eps:<8g} ->  (no roots found in
                                        the search box)")
    plt.show()

def cmd_transfer(args, params=None):
    factory = make_tf_factory()
    if params is None:
        params = _param_grid(args.eta, args.eps)
    fig, ax = plot_transfer_function_overlay(
        factory,
        params,
        omega_range=(args.omega_min, args.omega_max),
        log_omega=not args.linear,
        magnitude_db=args.db,
        title="Chapter 3 -- Frequency response of $G(s)$ (open loop)",
    )
    fig.savefig(args.output, dpi=150)
    print(f"Figure saved to: {args.output}")
    plt.show()

def _run_from_config():
    params = list(zip([ETA_1, ETA_2, ETA_3], [EPS_1, EPS_2, EPS_3]))
```

```
    print("(No command-line arguments: using the configuration "
          f"from the file -> command='{COMANDO}'")
    print(f" (eta, eps) = {params})\n")

    args = _parse_args([COMANDO])
    if COMANDO == "spectrum":
        cmd_spectrum(args, params=params)
    elif COMANDO == "transfer":
        cmd_transfer(args, params=params)
    else:
        raise SystemExit(f"COMANDO must be 'spectrum' or 'transfer', not {
                                                COMANDO!r}")

def main(argv=None):
    if argv is None and len(sys.argv) == 1:
        _run_from_config()
        return
    args = _parse_args(argv)
    if args.cmd == "spectrum":
        cmd_spectrum(args)
    elif args.cmd == "transfer":
        cmd_transfer(args)

if __name__ == "__main__":
    main()
```

Code 5

```
  #!/usr/bin/env python3
# -*- coding: utf-8 -*-
"""
lemma_3_7_winding_number.py
============================

"""

from __future__ import annotations

import argparse
import os
import sys

import numpy as np

import matplotlib
if os.environ.get("DISPLAY", "") == "" and sys.platform.startswith("linux")
                                        :
    matplotlib.use("Agg")
import matplotlib.pyplot as plt
from matplotlib.patches import Circle

LN2 = float(np.log(2.0))

def phi_gamma_exact(sigma, phi, gamma):
    """Gtilde(gamma,sigma,phi), identical to phi_gamma, exact formula,
                                          without
    taking gamma->0."""
    s = sigma + 1j * phi / gamma
    theta = np.sqrt(1 + gamma ** 2 * sigma + 1j * gamma * phi)
    return np.exp(s) * theta * (np.exp(s / theta) - np.exp(-s / theta)) + 1

def phi_gamma_asymptotic(sigma, phi, gamma):
```

```
    """Leading-order approximation (gamma -> 0^+),
        A(0,sigma,phi) = e^{2 sigma + phi^2/2},  B(phi) = 1 - e^{-phi^2/2}.
                                                """
    A0 = np.exp(2 * sigma + phi ** 2 / 2)
    B = 1 - np.exp(-phi ** 2 / 2)
    return np.exp(2j * phi / gamma) * A0 + B

def k_of_gamma(gamma):
    return int(np.floor(np.sqrt(2 * LN2) / (np.pi * gamma)))

def winding_number(z):
    """Winding number of a closed polygonal curve z around 0, via the
    sum of angles with atan2."""
    z1 = z
    z2 = np.roll(z, -1)
    cross = z1.real * z2.imag - z1.imag * z2.real
    dot = z1.real * z2.real + z1.imag * z2.imag
    dtheta = np.arctan2(cross, dot)
    return np.sum(dtheta) / (2 * np.pi)

def build_figure(gamma=0.2, delta=0.15, n_points=1500, output="
                                    lemma_3_7_winding_number.png"):
    k = k_of_gamma(gamma)
    phi_lo, phi_hi = k * np.pi * gamma, (k + 1) * np.pi * gamma
    sigma_lo, sigma_hi = -LN2 - delta, -LN2 + delta

    sigmas = np.linspace(sigma_lo, sigma_hi, n_points)
    phis = np.linspace(phi_lo, phi_hi, n_points)

    # Colors by side (match between the rectangle and its image curve)
    COLOR_PHI_HI = "#d62728"     # red    -- varphi = (k+1)pi*gamma
    COLOR_SIGMA_HI = "#ff7f0e"   # orange -- sigma = -ln2+delta
    COLOR_PHI_LO = "#1f77b4"     # blue   -- varphi = k*pi*gamma
    COLOR_SIGMA_LO = "#2ca02c"   # green  -- sigma = -ln2-delta

    edge1_dom = (sigmas, np.full_like(sigmas, phi_hi))          # varphi=
                                        phi_hi, sigma incr.
    edge2_dom = (np.full_like(phis, sigma_hi), phis[::-1])      # sigma=
                                        sigma_hi, varphi decr.
    edge3_dom = (sigmas[::-1], np.full_like(sigmas, phi_lo))    # varphi=
                                        phi_lo, sigma decr.
    edge4_dom = (np.full_like(phis, sigma_lo), phis)            # sigma=
                                        sigma_lo, varphi incr.

    def image_exact(dom):
        sigma_arr, phi_arr = dom
        return phi_gamma_exact(sigma_arr, phi_arr, gamma)

    def image_asym(dom):
        sigma_arr, phi_arr = dom
        return phi_gamma_asymptotic(sigma_arr, phi_arr, gamma)

    edge1_img, edge2_img, edge3_img, edge4_img = map(image_exact, [
                                        edge1_dom, edge2_dom, edge3_dom,
                                        edge4_dom])
    edge1_a, edge2_a, edge3_a, edge4_a = map(image_asym, [edge1_dom,
                                        edge2_dom, edge3_dom, edge4_dom])

    full_curve = np.concatenate([edge1_img, edge2_img, edge3_img, edge4_img
                                        ])
    full_curve_a = np.concatenate([edge1_a, edge2_a, edge3_a, edge4_a])
    w = winding_number(full_curve)
```

```
w_a = winding_number(full_curve_a)

err1 = np.max(np.abs(edge1_img - edge1_a))
err2 = np.max(np.abs(edge2_img - edge2_a))
err3 = np.max(np.abs(edge3_img - edge3_a))
err4 = np.max(np.abs(edge4_img - edge4_a))

# ------------------------------------------------------------------
fig, (ax_dom, ax_img) = plt.subplots(1, 2, figsize=(13, 6))

# ---- Left panel: the rectangle Omega^gamma (sigma horizontal) ----
ax_dom.plot(*edge1_dom, color=COLOR_PHI_HI, linewidth=3,
            label=r"$\varphi=(k{+}1)\pi\gamma$")
ax_dom.plot(*edge2_dom, color=COLOR_SIGMA_HI, linewidth=3,
            label=r"$\sigma=-\ln2+\delta$")
ax_dom.plot(*edge3_dom, color=COLOR_PHI_LO, linewidth=3,
            label=r"$\varphi=k\pi\gamma$")
ax_dom.plot(*edge4_dom, color=COLOR_SIGMA_LO, linewidth=3,
            label=r"$\sigma=-\ln2-\delta$")

def add_arrow(ax, x, y, color, frac=0.5):
    n = len(x)
    i = int(frac * (n - 1))
    j = min(i + 2, n - 1)
    ax.annotate("", xy=(x[j], y[j]), xytext=(x[i], y[i]),
                arrowprops=dict(arrowstyle="-|>", color=color, lw=2,
                                                    mutation_scale=22
                                                    ))

for dom, color in [(edge1_dom, COLOR_PHI_HI), (edge2_dom,
                                    COLOR_SIGMA_HI),
                   (edge3_dom, COLOR_PHI_LO), (edge4_dom,
                                                    COLOR_SIGMA_LO
                                                    )]:
    add_arrow(ax_dom, *dom, color)

ax_dom.scatter([sigma_lo, sigma_hi, sigma_lo, sigma_hi],
               [phi_lo, phi_lo, phi_hi, phi_hi],
               color="black", zorder=5, s=25)
ax_dom.axvline(-LN2, color="gray", linestyle=":", linewidth=1)
ax_dom.set_xlabel(r"$\sigma$")
ax_dom.set_ylabel(r"$\varphi$")
ax_dom.set_title(r"Domain $\Omega^\gamma=(-\ln2-\delta,-\ln2+\delta)\
                                    times"
                 r"(k\pi\gamma,(k{+}1)\pi\gamma)$, clockwise" + "\n"
                 rf"$\gamma={gamma:g}$, $\delta={delta:g}$, $k(\gamma)
                                                    ={k}$")
ax_dom.legend(loc="upper left", fontsize=8.5, framealpha=0.9)
ax_dom.grid(alpha=0.3)
margin_x = 0.5 * delta
margin_y = 0.12 * (phi_hi - phi_lo)
ax_dom.set_xlim(sigma_lo - margin_x, sigma_hi + margin_x)
ax_dom.set_ylim(phi_lo - margin_y, phi_hi + margin_y)

# ---- Right panel: the image under phi_gamma
ax_img.plot(edge1_img.real, edge1_img.imag, color=COLOR_PHI_HI,
                                    linewidth=2.4,
            label=r"image of $\varphi=(k{+}1)\pi\gamma$ (exact)")
ax_img.plot(edge2_img.real, edge2_img.imag, color=COLOR_SIGMA_HI,
                                    linewidth=2.4,
            label=r"image of $\sigma=-\ln2+\delta$ (exact)")
```

```
ax_img.plot(edge3_img.real, edge3_img.imag, color=COLOR_PHI_LO,
                                            linewidth=2.4,
            label=r"image of $\varphi=k\pi\gamma$ (exact)")
ax_img.plot(edge4_img.real, edge4_img.imag, color=COLOR_SIGMA_LO,
                                            linewidth=2.4,
            label=r"image of $\sigma=-\ln2-\delta$ (exact)")

for img_a, color in [(edge1_a, COLOR_PHI_HI), (edge2_a, COLOR_SIGMA_HI)
                                            ,
                     (edge3_a, COLOR_PHI_LO), (edge4_a, COLOR_SIGMA_LO
                                                )]:
    ax_img.plot(img_a.real, img_a.imag, color=color, linewidth=1.3,
                linestyle=(0, (2, 2)), alpha=0.65)

for img, color in [(edge1_img, COLOR_PHI_HI), (edge2_img,
                                            COLOR_SIGMA_HI),
                   (edge3_img, COLOR_PHI_LO), (edge4_img,
                                                        COLOR_SIGMA_LO
                                                        )]:
    add_arrow(ax_img, img.real, img.imag, color)

ax_img.scatter([0], [0], color="black", marker="x", s=90, zorder=6,
                                            label=r"origin")
ax_img.axhline(0, color="gray", linewidth=0.6)
ax_img.axvline(0, color="gray", linewidth=0.6)
ax_img.set_xlabel(r"$\Re\,\phi_\gamma$")
ax_img.set_ylabel(r"$\Im\,\phi_\gamma$")
ax_img.set_title(r"Image $\phi_\gamma(\partial\Omega^\gamma)$,
                                            clockwise"
                 + "\n" + rf"winding number: ${w:.4f}$")
ax_img.legend(loc="upper right", fontsize=7.5, framealpha=0.9)
ax_img.grid(alpha=0.3)
ax_img.set_aspect("equal", adjustable="box")

R_big = 0.5 * np.exp(2 * delta)
R_small = 0.5 * np.exp(-2 * delta)
for R, color in [(R_big, COLOR_SIGMA_HI), (R_small, COLOR_SIGMA_LO)]:
    circ = Circle((0.5, 0), R, fill=False, edgecolor=color, linewidth=0
                                            .8,
                  linestyle=":", alpha=0.35)
    ax_img.add_patch(circ)

info = (
    r"Solid = exact formula $\phi_\gamma(\sigma,\varphi)$ (before
                                            taking the limit $\gamma\to0
                                            ^+$)"
    "     "
    r"Dotted = leading-order asymptotic approximation ($\gamma\to0^+$)"
    "\n"
    rf"Max. error (sup-norm) exact$-$asymptotic: "
    rf"$\varphi$hi={err1:.4f}, $\sigma$hi={err2:.4f}, $\varphi$lo={err3
                                            :.4f}, $\sigma$lo={err4:.4f}"
    "\n"
    rf"Winding number (standard mathematical sign convention,
                                            counterclockwise$=+1$): "
    rf"exact$={w:.6f}$, asymptotic$={w_a:.6f}$  "
    r"$\Rightarrow$ 1 clockwise loop, exactly as described in the proof
                                            "
    + (r" ($\checkmark$ Lemma 3.7)" if abs(w + 1) < 1e-2 else "")
)
fig.text(0.5, -0.04, info, ha="center", va="top", fontsize=8.5,
```

```
            bbox=dict(boxstyle="round", facecolor="whitesmoke", edgecolor
                                                    ="gray"))

    fig.suptitle("Proof of Lemma 3.7 âĂŤ domain of the topological degree
                                        and its image under "
                 r"$\phi_\gamma$ (clockwise traversal)", fontsize=13, y=1.
                                                    03)
    fig.tight_layout()
    fig.savefig(output, dpi=160, bbox_inches="tight")
    print(f"Figure saved to: {output}")
    print(f"\nk(gamma) = {k}")
    print(f"sigma range: [{sigma_lo:.6f}, {sigma_hi:.6f}]")
    print(f"phi range:   [{phi_lo:.6f}, {phi_hi:.6f}]")
    print(f"Large-circle radius (sigma=-ln2+delta): R_+ = (1/2)e^(2 delta)
                                        = {R_big:.6f}")
    print(f"Small-circle radius (sigma=-ln2-delta): R_- = (1/2)e^(-2 delta)
                                         = {R_small:.6f}")
    print(f"Winding number, standard sign convention (EXACT curve):      {w
                                        :.8f}")
    print(f"Winding number, standard sign convention (asymptotic model): {
                                        w_a:.8f}")
    print("(-1 = one clockwise loop, exactly as described in the proof of
                                        Lemma 3.7)")
    print(f"Max |exact - asymptotic| error per side: phi_hi={err1:.6f}
                                        sigma_hi={err2:.6f}  "
          f"phi_lo={err3:.6f}  sigma_lo={err4:.6f}")
    return fig, w, dict(phi_hi=err1, sigma_hi=err2, phi_lo=err3, sigma_lo=
                                        err4)

def _parse_args(argv=None):
    p = argparse.ArgumentParser(description="Figure for the proof of Lemma
                                        3.7 (topological degree).")
    p.add_argument("--gamma", type=float, default=0.2, help="Parameter
                                        gamma = sqrt(eta).")
    p.add_argument("--delta", type=float, default=0.15, help="Half-width
                                        delta in sigma (must be in (0,
                                        ln2)).")
    p.add_argument("--n-points", type=int, default=1500)
    p.add_argument("--output", type=str, default="lemma_3_7_winding_number.
                                        png")
    return p.parse_args(argv)

def main(argv=None):
    args = _parse_args(argv)
    if not (0 < args.delta < LN2):
        raise SystemExit(f"--delta must be in (0, ln 2) = (0, {LN2:.6f})")
    build_figure(gamma=args.gamma, delta=args.delta, n_points=args.n_points
                                        , output=args.output)
    plt.show()

if __name__ == "__main__":
    main()
```

A.3 Codes for Chapter 5

Code 6

```
#!/usr/bin/env python3
# -*- coding: utf-8 -*-
"""
biharmonic_core.py
```

```
====================

"""

import mpmath as mp

def quartic_roots_mp(s, eta, eps):
    """The 4 roots of eta*lambda^4 + (1+eps)*lambda + s = 0."""
    eta = mp.mpf(eta)
    eps = mp.mpf(eps)
    c = 1 + eps
    return mp.polyroots([eta, 0, 0, c, s], maxsteps=400, extraprec=400)

def f_eta_eps_mp(s, eta, eps):
    """Transfer function of a single biharmonic channel, f_{eta,eps}(s).

    Special case eta=0 (no diffusion): the quartic degenerates to the
                                          single
    branch lambda=-s/(1+eps) (pure transport at speed 1+eps), giving
    f_{0,eps}(s) = e^{-s/(1+eps)}."""
    if eta == 0:
        c = 1 + mp.mpf(eps)
        return mp.exp(-s / c)

    c = 1 + mp.mpf(eps)
    l = quartic_roots_mp(s, eta, eps)

    # Classify: l1 = finite branch (closest to -s/(1+eps));
    # among the remaining 3, the one with Re<0 is the layer at x=0 (l2),
                                          the 2
    # with Re>0 are the layer at x=1, l3 and l4.
    l_sorted = sorted(l, key=lambda z: abs(z + s / c))
    l1 = l_sorted[0]
    rest = sorted(l_sorted[1:], key=lambda z: z.real)
    l2 = rest[0]
    l3, l4 = rest[1], rest[2]

    # 4x4 system in the basis adapted to the boundary layers.
    M = mp.matrix(4, 4)
    M[0, 0] = 1;            M[0, 1] = 1;            M[0, 2] = mp.e ** (-l3)
                                          ;              M[0, 3] = mp.e **
                                          (-l4)
    M[1, 0] = l1 ** 2;       M[1, 1] = l2 ** 2;       M[1, 2] = (l3 ** 2) *
                                           mp.e ** (-l3);   M[1, 3] = (l4
                                          ** 2) * mp.e ** (-l4)
    M[2, 0] = (l1 ** 2) * mp.e ** l1; M[2, 1] = (l2 ** 2) * mp.e ** l2; M[2
                                          , 2] = l3 ** 2;   M[2, 3] = l4 **
                                           2
    M[3, 0] = (l1 ** 3) * mp.e ** l1; M[3, 1] = (l2 ** 3) * mp.e ** l2; M[3
                                          , 2] = l3 ** 3;   M[3, 3] = l4 **
                                           3

    rhs = mp.matrix([1, 0, 0, 0])
    A1, A2, B3, B4 = mp.lu_solve(M, rhs)
    return A1 * mp.e ** l1 + A2 * mp.e ** l2 + B3 + B4

def char_closed_loop_mp_factory(eta, eps):
    """h(s) = f^2(s) - f(s) e^{-s} - 1   closed loop with delay
    predictor, U(t) = -Y(t-1))."""
    def h(s):
        f = f_eta_eps_mp(s, eta, eps)
        return f * f - f * mp.exp(-s) - 1
```

```
    return h

def char_open_loop_mp_factory(eta, eps):
    """h(s) = f^2(s) - 1   open loop: simple reflection, without the
    delay predictor."""
    def h(s):
        f = f_eta_eps_mp(s, eta, eps)
        return f * f - 1
    return h
```

Code 7

```
  #!/usr/bin/env python3
# -*- coding: utf-8 -*-
"""
open_loop_spectrum.py
=======================

"""

from biharmonic_core import char_open_loop_mp_factory
from char_eq_utils import LN2, plot_spectrum_overlay
import matplotlib.pyplot as plt

ETA_LIST = [0.1, 0.01, 0.001]
EPS_LIST = [0.0]

RE_MIN, RE_MAX = -6.0, 1.0
IM_MAX = 20.0
N_RE, N_IM = 16, 16
DPS = 30  # mpmath precision digits

def main():
    params = [(eta, eps) for eta in ETA_LIST for eps in EPS_LIST]

    fig, ax, info = plot_spectrum_overlay(
        char_open_loop_mp_factory,
        params,
        re_range=(RE_MIN, RE_MAX),
        im_range=(0.0, IM_MAX),
        n_re=N_RE,
        n_im=N_IM,
        dps=DPS,
        title="Biharmonic open-loop spectrum",
        verbose=True,
    )

    print("\nSummary (maximum spectral abscissa Re(s) found, compare with -
                                        ln(2) = "
          f"{-LN2:.6f}):")
    for eta, eps, roots in info:
        if roots:
            print(f"  eta={eta:<8g} eps={eps:<8g} ->  max Re(s) = {roots[0
                                                    ].real:.6f}")
        else:
            print(f"  eta={eta:<8g} eps={eps:<8g} ->  (no roots found in
                                                    the search box)")

    plt.show()

if __name__ == "__main__":
    main()
```

Code 8

```
  #!/usr/bin/env python3
# -*- coding: utf-8 -*-
"""
closed_loop_spectrum.py
=========================

"""

from biharmonic_core import char_closed_loop_mp_factory
from char_eq_utils import LN2, plot_spectrum_overlay
import matplotlib.pyplot as plt

ETA_LIST = [0.1, 0.01, 0.001]
EPS_LIST = [0.0]

RE_MIN, RE_MAX = -6.0, 1.0
IM_MAX = 20.0
N_RE, N_IM = 16, 16
DPS = 30  # mpmath precision digits

def main():
    params = [(eta, eps) for eta in ETA_LIST for eps in EPS_LIST]

    fig, ax, info = plot_spectrum_overlay(
        char_closed_loop_mp_factory,
        params,
        re_range=(RE_MIN, RE_MAX),
        im_range=(0.0, IM_MAX),
        n_re=N_RE,
        n_im=N_IM,
        dps=DPS,
        title="Biharmonic closed-loop spectrum",
        verbose=True,
    )

    print("\nSummary (maximum spectral abscissa Re(s) found, compare with -
                                        ln(2) = "
          f"{-LN2:.6f}):")
    for eta, eps, roots in info:
        if roots:
            print(f"  eta={eta:<8g} eps={eps:<8g} ->  max Re(s) = {roots[0
                                                    ].real:.6f}")
        else:
            print(f"  eta={eta:<8g} eps={eps:<8g} ->  (no roots found in
                                                    the search box)")

    plt.show()

if __name__ == "__main__":
    main()
```

Code 9

```
  #!/usr/bin/env python3
# -*- coding: utf-8 -*-
"""
robustness_spectrum.py
=========================

"""
```

```
from biharmonic_core import char_closed_loop_mp_factory
from char_eq_utils import LN2, plot_spectrum_overlay
import matplotlib.pyplot as plt

ETA_1, ETA_2, ETA_3 = 0.02, 0.02, 0.0
EPS_1, EPS_2, EPS_3 = 0.0, 0.1, 0.1

RE_MIN, RE_MAX = -6.0, 1.0
IM_MAX = 20.0
N_RE, N_IM = 16, 16
DPS = 30  # mpmath precision digits

def main():
    params = list(zip([ETA_1, ETA_2, ETA_3], [EPS_1, EPS_2, EPS_3]))
    print(f"Combinations (eta, eps) to plot: {params}\n")

    fig, ax, info = plot_spectrum_overlay(
        char_closed_loop_mp_factory,
        params,
        re_range=(RE_MIN, RE_MAX),
        im_range=(0.0, IM_MAX),
        n_re=N_RE,
        n_im=N_IM,
        dps=DPS,
        title="Biharmonic closed-loop spectrum: robustness under speed
                                                perturbation",
        verbose=True,
    )

    print("\nSummary (maximum spectral abscissa Re(s) found, compare with -
                                            ln(2) = "
          f"{-LN2:.6f}):")
    for eta, eps, roots in info:
        if roots:
            print(f"  eta={eta:<8g} eps={eps:<8g} ->  max Re(s) = {roots[0
                                                    ].real:.6f}")
        else:
            print(f"  eta={eta:<8g} eps={eps:<8g} ->  (no roots found in
                                                    the search box)")

    plt.show()

if __name__ == "__main__":
    main()
```

Code 10

```
  """
winding_number.py

"""
import mpmath as mp
import numpy as np
import matplotlib.pyplot as plt

mp.mp.dps = 30

SIDE_COLORS = {
    "bottom": "tab:blue",
    "right": "tab:orange",
    "top": "tab:green",
    "left": "tab:purple",
```

```
}
SIDE_LABELS_ES = {
    "bottom": r"bottom side ($\omega=k\pi$)",
    "right": r"right side ($\sigma=\sigma_*+\delta_0$)",
    "top": r"top side ($\omega=(k{+}1)\pi$)",
    "left": r"left side ($\sigma=\sigma_*-\delta_0$)",
}

def f_eta(s, eta):
    coeffs = [eta, 0, 0, 1, s]
    l = mp.polyroots(coeffs, maxsteps=300, extraprec=300)
    M = mp.matrix(4, 4)
    for k in range(4):
        M[0, k] = 1
    for k in range(4):
        M[1, k] = l[k] ** 2
    for k in range(4):
        M[2, k] = (l[k] ** 2) * mp.e ** l[k]
    for k in range(4):
        M[3, k] = (l[k] ** 3) * mp.e ** l[k]
    A = mp.lu_solve(M, mp.matrix([1, 0, 0, 0]))
    return sum(A[k] * mp.e ** l[k] for k in range(4))

def G(s, eta):
    f = f_eta(s, eta)
    return f ** 2 - f * mp.e ** (-s) - 1

def boundary_points_by_side(sig_lo, sig_hi, w_lo, w_hi, n_per_side=40):
    sides = {}
    sides["bottom"] = [mp.mpc(sig_lo + t * (sig_hi - sig_lo), w_lo)
                       for t in np.linspace(0, 1, n_per_side, endpoint=
                                            False)]
    sides["right"] = [mp.mpc(sig_hi, w_lo + t * (w_hi - w_lo))
                      for t in np.linspace(0, 1, n_per_side, endpoint=
                                           False)]
    sides["top"] = [mp.mpc(sig_hi - t * (sig_hi - sig_lo), w_hi)
                    for t in np.linspace(0, 1, n_per_side, endpoint=False)
                                                        ]
    sides["left"] = [mp.mpc(sig_lo, w_hi - t * (w_hi - w_lo))
                     for t in np.linspace(0, 1, n_per_side, endpoint=False
                                                         )]
    return sides

def winding_number(eta, sig_lo, sig_hi, w_lo, w_hi, n_per_side=40):
    sides = boundary_points_by_side(sig_lo, sig_hi, w_lo, w_hi, n_per_side)
    order = ["bottom", "right", "top", "left"]
    pts = []
    for name in order:
        pts.extend(sides[name])
    pts.append(pts[0])  # close the loop

    vals = [complex(G(s, eta)) for s in pts]
    total = 0.0
    for i in range(len(vals) - 1):
        v1, v2 = vals[i], vals[i + 1]
        total += np.angle(v2 / v1)
    min_mod = min(abs(v) for v in vals)

    vals_by_side = {}
    pts_by_side = {}
    for i, name in enumerate(order):
        start = i * n_per_side
```

```
        end = start + n_per_side + 1
        vals_by_side[name] = vals[start:end]
        pts_by_side[name] = pts[start:end]

    return total / (2 * np.pi), min_mod, pts_by_side, vals_by_side

def plot_winding(eta, k_band):
    ln2 = float(np.log(2))
    d0 = 0.35
    w_lo, w_hi = k_band * np.pi, (k_band + 1) * np.pi
    wn, mm, pts_by_side, vals_by_side = winding_number(
        eta, -ln2 - d0, -ln2 + d0, w_lo, w_hi, n_per_side=80)

    fig, axes = plt.subplots(1, 2, figsize=(13, 6))

    # ---------------- left panel: the rectangle, side by side
    ax = axes[0]
    for name, pts in pts_by_side.items():
        sig = [float(p.real) for p in pts]
        om = [float(p.imag) for p in pts]
        ax.plot(sig, om, color=SIDE_COLORS[name], lw=3,
                label=SIDE_LABELS_ES[name], solid_capstyle="round")

    ax.axvline(-ln2, color="k", ls=":", lw=1, label=r"$\sigma=-\ln2$")
    ax.axhline((k_band + 0.5) * np.pi, color="k", ls="--", lw=1,
               label=r"$\omega=(k+\frac{1}{2})\pi$")
    ax.plot(-ln2, (k_band + 0.5) * np.pi, "r*", markersize=15, zorder=5)
    ax.set_xlabel(r"$\sigma$")
    ax.set_ylabel(r"$\omega$")
    ax.set_title(fr"Rectangle $\Omega_\gamma$ ($\eta={eta:.0e}$, band $k={
                                            k_band}$)")
    ax.legend(fontsize=6.5, loc="lower right", framealpha=0.85)
    ax.grid(alpha=0.3)

    # ---------------- right panel: the image curve, side by side
    ax = axes[1]
    all_re, all_im = [], []
    for name, vals in vals_by_side.items():
        re = [v.real for v in vals]
        im = [v.imag for v in vals]
        all_re += re
        all_im += im
        ax.plot(re, im, color=SIDE_COLORS[name], lw=2.4, zorder=3,
                label=SIDE_LABELS_ES[name])
        mid = len(re) // 2
        if mid + 1 < len(re):
            ax.annotate("", xy=(re[mid + 1], im[mid + 1]), xytext=(re[mid],
                                                    im[mid]),
                        arrowprops=dict(arrowstyle="->", color=SIDE_COLORS[
                                                        name], lw=2),
                        zorder=4)
        ax.plot(re[0], im[0], "o", color=SIDE_COLORS[name], markersize=7,
                markeredgecolor="k", zorder=5)

    ax.plot(0, 0, "r*", markersize=18, zorder=6, label="origin")

    ax.set_xlabel(r"$\Re(\mathcal{G})$")
    ax.set_ylabel(r"$\Im(\mathcal{G})$")
    ax.set_title(f"Image $\\mathcal{{G}}(\\partial Q_\\gamma)$, \n"
                 f"winding = {wn:+.3f},  min$|\\mathcal{{G}}|$={mm:.3f}")
    ax.legend(fontsize=6.5, loc="lower right", framealpha=0.85)
    ax.grid(alpha=0.3)
```

```
    all_re = np.array(all_re)
    all_im = np.array(all_im)
    pad_x = 0.1 * (all_re.max() - all_re.min())
    pad_y = 0.1 * (all_im.max() - all_im.min())
    ax.set_xlim(all_re.min() - pad_x, all_re.max() + pad_x)
    ax.set_ylim(all_im.min() - pad_y, all_im.max() + pad_y)
    ax.set_aspect("equal")

    plt.tight_layout()
    plt.show()
    return wn, mm

if __name__ == "__main__":
    ln2 = float(np.log(2))
    d0 = 0.35

    print("Phase quantization: omega ~ (k+1/2)*pi at the roots")
    data = [(4.701970401, 0.001), (7.8762704, 0.0002),
            (11.008353, 0.00005), (17.30209, 0.00001)]
    for omega, eta in data:
        k_half = round(omega / np.pi - 0.5)
        pred = (k_half + 0.5) * np.pi
        print(f"  eta={eta:9.6f}  omega={omega:9.4f}  "
              f"(k+1/2)*pi={pred:9.4f}  |diff|={abs(omega - pred):.4f}")
    print()

    print("Winding number of G around the resonant rectangle (expect +1):")
    for eta_val, k_band in [(1e-3, 1), (2e-4, 2), (5e-5, 3)]:
        w_lo, w_hi = k_band * np.pi, (k_band + 1) * np.pi
        wn, mm, _, _ = winding_number(eta_val, -ln2 - d0, -ln2 + d0, w_lo,
                                      w_hi)
        print(f"  eta={eta_val:9.6f}  band omega=({w_lo:.3f},{w_hi:.3f})  "
              f"winding={wn:+.4f}  min|G| on boundary={mm:.3e}")

    plot_winding(1e-3, 1)
```

Bibliography

[AA17a] Henrik Anfinsen and Ole Morten Aamo. "Adaptive Output-Feedback Stabilization of Linear 2×2 Hyperbolic Systems Using Anti-Collocated Sensing and Control". *Systems & Control Letters* 104 (2017), pp. 86–94.

[AA17b] Henrik Anfinsen and Ole Morten Aamo. "Disturbance Rejection in General Heterodirectional 1-D Linear Hyperbolic Systems Using Collocated Sensing and Control". *Automatica* 76 (2017), pp. 230–242.

[Aam13] Ole Morten Aamo. "Disturbance Rejection in 2×2 Linear Hyperbolic Systems". *IEEE Transactions on Automatic Control* 58.5 (2013), pp. 1095–1106.

[AB22] Jean Auriol and Delphine Bresch-Pietri. "Robust State-Feedback Stabilization of an Underactuated Network of Interconnected $n + m$ Hyperbolic PDE Systems". *Automatica* 136 (2022), p. 110040. DOI: 10.1016/j.automatica.2021.110040.

[ÅM08] Karl Johan Åström and Richard M. Murray. *Feedback Systems: An Introduction for Scientists and Engineers*. 1st. Princeton, NJ: Princeton University Press, 2008.

[Aur+18a] Jean Auriol et al. "Delay-Robust Control Design for Two Heterodirectional Linear Coupled Hyperbolic PDEs". *IEEE Transactions on Automatic Control* 63.10 (2018), pp. 3551–3557. DOI: 10.1109/TAC.2018.2798818.

[Aur+18b] Jean Auriol et al. "Delay-Robust Stabilization of a Hyperbolic PDE–ODE System". *Automatica* 95 (2018), pp. 494–502. DOI: 10.1016/j.automatica.2018.06.033.

[BC16] Georges Bastin and Jean-Michel Coron. *Stability and Boundary Stabilization of 1-D Hyperbolic Systems*. Vol. 88. Progress in Nonlinear Differential Equations and Their Applications. Cham: Birkhäuser, 2016.

[BCH23] Georges Bastin, Jean-Michel Coron, and Amaury Hayat. "Diffusion and robustness of boundary feedback stabilization of hyperbolic systems". *Mathematics of Control, Signals, and Systems* 35.1 (2023), pp. 159–185. DOI: 10.1007/s00498-022-00335-0.

[BCH25a] Georges Bastin, Jean-Michel Coron, and Amaury Hayat. "A necessary and sufficient robustness condition for boundary observer-controllers that stabilize 2×2 hyperbolic systems". Preprint, HAL hal-05406888. 2025. URL: https://hal.science/hal-05406888.

[BCH25b] Georges Bastin, Jean-Michel Coron, and Amaury Hayat. "The usefulness of viscosity for the robustness of boundary feedback control of an unstable fluid flow system". *Automatica* 173 (2025). DOI: 10.1016/j.automatica.2024.112048.

[BK14] Pauline Bernard and Miroslav Krstic. "Adaptive output-feedback stabilization of non-local hyperbolic PDEs". *Automatica* 50.10 (2014), pp. 2692–2699. DOI: 10.1016/j.automatica.2014.09.001.

[Bod45] Hendrik W. Bode. *Network Analysis and Feedback Amplifier Design*. New York: D. Van Nostrand Company, 1945.

[CBd08] Jean-Michel Coron, Georges Bastin, and Brigitte d'Andréa-Novel. "Dissipative Boundary Conditions for One-Dimensional Nonlinear Hyperbolic Systems". *SIAM Journal on Control and Optimization* 47.3 (2008), pp. 1460–1498. DOI: 10.1137/070706847.

[CM09] Ruth F. Curtain and Kirsten A. Morris. "Transfer functions of distributed parameter systems: A tutorial". *Automatica* 45.5 (2009), pp. 1101–1116. DOI: 10.1016/j.automatica.2009.01.008.

[Cor07] Jean-Michel Coron. *Control and Nonlinearity*. Vol. 136. Mathematical Surveys and Monographs. Providence, RI: American Mathematical Society, 2007.

[CW13] Jean-Michel Coron and Zhiqiang Wang. "Output Feedback Stabilization for a Scalar Conservation Law with a Nonlocal Velocity". *SIAM Journal on Mathematical Analysis* 45.5 (Jan. 2013), pp. 2646–2665. DOI: 10.1137/120902203. URL: https://doi.org/10.1137/120902203.

[CZ95] Ruth F. Curtain and Hans Zwart. *An Introduction to Infinite-Dimensional Linear Systems Theory*. Vol. 21. Texts in Applied Mathematics. New York: Springer, 1995.

[Dat88] Richard Datko. "Not All Feedback Stabilized Hyperbolic Systems are Robust with Respect to Small Time Delays in Their Feedbacks". *SIAM Journal on Control and Optimization* 26.3 (May 1988), pp. 697–713. DOI: 10.1137/0326040. URL: https://doi.org/10.1137/0326040.

[DFT92] John C. Doyle, Bruce A. Francis, and Allen R. Tannenbaum. *Feedback Control Theory*. Reprinted, Dover Publications, Mineola, NY, 2009. New York: Macmillan Publishing Co., 1992.

[DP08] Valérie Dos Santos Martins and Christophe Prieur. "Boundary Control of Open Channels With Numerical and Experimental Validations". *IEEE Transactions on Control Systems Technology* 16.6 (2008), pp. 1252–1264.

[EN00] Klaus-Jochen Engel and Rainer Nagel. *One-Parameter Semigroups for Linear Evolution Equations*. Vol. 194. Graduate Texts in Mathematics. New York: Springer-Verlag, 2000.

[FG95] Irene Fonseca and Wilfrid Gangbo. *Degree Theory in Analysis and Applications*. Vol. 2. Oxford Lecture Series in Mathematics and its Applications. New York: The Clarendon Press, Oxford University Press, 1995.

[Fri14] Emilia Fridman. *Introduction to Time-Delay Systems: Analysis and Control*. Systems & Control: Foundations & Applications. Birkhäuser/Springer, 2014.

[GX07] Bao-Zhu Guo and Cheng-Zhong Xu. "The stabilization of a one-dimensional wave equation by boundary feedback with noncollocated observation". *IEEE Transactions on Automatic Control* 52.2 (2007), pp. 371–377. DOI: 10.1109/TAC.2006.890385.

[HS19] Amaury Hayat and Peipei Shang. "A quadratic Lyapunov function for Saint-Venant equations with arbitrary friction and space-varying slope". *Automatica* 100 (2019), pp. 52–60.

[HT22] Agus Hasan and Shu-Xia Tang. "Boundary Control of a Coupled Burgers' PDE–ODE System". *International Journal of Robust and Nonlinear Control* 32.10 (2022), pp. 5812–5836. DOI: 10.1002/rnc.6145.

[HV93] Jack K. Hale and Sjoerd M. Verduyn Lunel. *Introduction to Functional-Differential Equations*. Vol. 99. Applied Mathematical Sciences. New York: Springer-Verlag, 1993.

[Kai80] Thomas Kailath. *Linear Systems*. Englewood Cliffs, NJ: Prentice-Hall, 1980.

[Kat66] Tosio Kato. *Perturbation theory for linear operators*. Vol. Band 132. Die Grundlehren der mathematischen Wissenschaften. Springer-Verlag New York, Inc., New York, 1966, pp. xix+592.

[KK22] Iasson Karafyllis and Miroslav Krstic. "Spill-Free Transfer and Stabilization of Viscous Liquid". *IEEE Transactions on Automatic Control* 67.9 (2022), pp. 4585–4597.

[Krs+08] Miroslav Krstic et al. "Output-feedback stabilization of an unstable wave equation". *Automatica* 44.1 (2008), pp. 63–74. DOI: 10.1016/j.automatica.2007.05.012.

[Lee13] John M. Lee. *Introduction to smooth manifolds*. Second. Vol. 218. Graduate Texts in Mathematics. Springer, New York, 2013, pp. xvi+708. ISBN: 978-1-4419-9981-8.

[Lic08] Mark Lichtner. "Spectral mapping theorem for linear hyperbolic systems". *Proceedings of the American Mathematical Society* 136.6 (2008), pp. 2091–2101. DOI: 10.1090/S0002-9939-08-09181-8.

[MN07] Wim Michiels and Silviu-Iulian Niculescu. *Stability and Stabilization of Time-Delay Systems: An Eigenvalue-Based Approach*. Vol. 12. Advances in Design and Control. Philadelphia: SIAM, 2007.

[Mor20] Kirsten A. Morris. *Controller Design for Distributed Parameter Systems*. Communications and Control Engineering. Cham: Springer, 2020.

[Nyq32] Harry Nyquist. "Regeneration Theory". *Bell System Technical Journal* 11.1 (Jan. 1932), pp. 126–147. DOI: 10.1002/j.1538-7305.1932.tb02344.x.

[Oga10] Katsuhiko Ogata. *Modern Control Engineering*. 5th. Upper Saddle River, NJ: Prentice Hall, 2010.

[ZP17] Liguo Zhang and Christophe Prieur. "Necessary and sufficient conditions on the exponential stability of positive hyperbolic systems". *IEEE Transactions on Automatic Control* 62.7 (Jan. 2017), pp. 3610–3617. DOI: 10.1109/tac.2017.2661966. URL: https://doi.org/10.1109/tac.2017.2661966.